Theory of Discrete and Ultradiscrete Integrable Finite Lattices Associated with Orthogonal Polynomials and Its Applications(Dissertation_全文)

AUTHOR(S):

Maeda, Kazuki

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by

Kazuki MAEDA

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Chapter 1

Introduction

In this thesis, we study the theory and applications of discrete and ultradiscrete integrable systems with non-periodic finite lattice boundary condition. We reformulate the theory of spectral transformations for orthogonal polynomials and their generalizations, and derive and analyze discrete integrable semi-infinite and finite lattices using this theory. As applications of the theory, we study a connection between the box–ball systems and the derived finite lattices, and discuss an extension of the connection. As more practical applications for engineering, we develop a generalized eigenvalue algorithm for tridiagonal matrix pencils based on the derived finite lattice equations.

In the following sections, we give a brief exposition of the basic concepts appearing in this thesis and describe the outline.

1.1 Integrable systems

In the theory of dynamical systems, the following classical theorem is well known.

Theorem 1.1 (Liouville–Arnol'd [5]). *Let us consider a Hamiltonian system with N degrees of freedom:*

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \dots, N.$$

If N first integrals $F_1, F_2, \dots, F_N$ are in involution:

$$\{F_i, F_j\} = \sum_{k=1}^{N} \left(\frac{\partial F_i}{\partial q_k} \frac{\partial F_j}{\partial p_k} - \frac{\partial F_j}{\partial q_k} \frac{\partial F_i}{\partial p_k} \right) = 0,$$

then the system is ***completely integrable*** *by quadrature.*

The Liouville–Arnol'd theorem states on the integrability of finite dimensional dynamical systems. For infinite dimensional systems, although there is no rigorous definition, the concept of integrability is also used; if a nonlinear system has (nontrivial) particular solutions written down explicitly, one may say the system is integrable. It is empirically known that integrable systems share certain characteristics: the existence of infinitely many conserved quantities, Lax representation [43], determinant solutions, and so on.

1.1.1 Continuous integrable systems

Historically, the theory of infinite dimensional integrable systems began with the discovery of *solitons*, solitary waves whose shapes are preserved while they propagate. The first soliton equation was discovered in 1895. It is now called the *Korteweg–de Vries (KdV) equation* [41]:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u = u(t, x), \tag{1.1}$$

where $t \in \mathbb{R}$ is the time variable and $x \in \mathbb{R}$ is the space variable.

Gardner *et al*. [14] first integrated the KdV equation (1.1) by use of inverse scattering method in 1967. On the other hand, Hirota [22] developed a direct method for solving the KdV equation (1.1) in 1971. He considered the following variable transformation:

$$u(t, x) = -2\frac{\partial^2}{\partial x^2} \ln f(t, x).$$

Then, the KdV equation (1.1) is transformed into the *bilinear equation*

$$f\frac{\partial^2 f}{\partial t \partial x} - \frac{\partial f}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial^4 f}{\partial x^4} f - 4\frac{\partial^3 f}{\partial x^3}\frac{\partial f}{\partial x} + 3\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial x^2} = 0. \tag{1.2}$$

By using the bilinear equation (1.2), he proved that the KdV equation (1.1) has *N-soliton solutions* written by the determinant:

$$f(t, x) = \left|M_{i,j}(t, x)\right|_{1\le i,j\le N},$$
$$M_{i,j}(t, x) := \delta_{i,j} + \frac{2(p_i p_j)^{1/2}}{p_i + p_j} \exp\left(\frac{\xi_i(t, x) + \xi_j(t, x)}{2}\right), \quad \xi_i(t, x) := p_i x - p_i^3 t - \eta_i,$$

where $\delta_{i,j}$ denotes the Kronecker delta and p_i and η_i are arbitrary constants that determine the amplitude and phase, respectively. Figure 1.1 shows an example of the 2-soliton interaction.

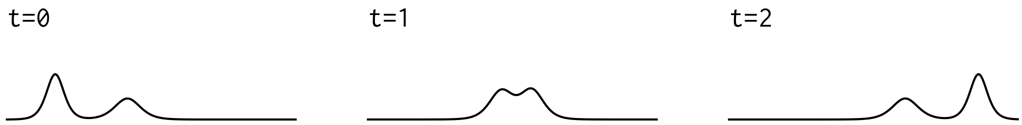


Figure 1.1: Example of the 2-soliton interaction of the KdV equation.

One may view Hirota's method as an analogue of the linearization of nonlinear ordinary differential equations, but its underlying mathematical structure is more complicated. The integrability of the bilinear equation (1.2) is not obvious. In general, even if one can transform a nonlinear differential equation into a bilinear equation, the existence of solutions such as the ones above is not assured. In 1983, Sato and Sato [63] studied bilinear equations of the *Kadomtsev–Petviashvili (KP) hierarchy*, which contains the KdV equation (1.1). They clarified that the bilinear equation (1.2) is one of the relations on the infinite dimensional Grassmannian called *Plücker relations*.

In 1968, Miura [49] considered the *modified KdV (mKdV) equation*

$$\frac{\partial v}{\partial t} - 6v^2\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0, \quad v = v(t, x), \tag{1.3}$$

and the following variable transformation

$$u = v^2 + \frac{\partial v}{\partial x}. \tag{1.4}$$

Substituting (1.4) into the KdV equation (1.1), we obtain

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \left(2v + \frac{\partial}{\partial x}\right)\left(\frac{\partial v}{\partial t} - 6v^2\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3}\right).$$

Hence, (1.4) gives a transformation from a solution of the mKdV equation (1.3) to a solution of the KdV equation (1.1). This type of transformation is called the *Miura type transformation*.

Another typical soliton equation is the *Toda lattice*:

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = \mathrm{e}^{q_{n-1}-q_n} - \mathrm{e}^{q_n - q_{n+1}}, \quad q_n = q_n(t), \tag{1.5}$$

where $n \in \mathbb{Z}$ is the space variable and $t \in \mathbb{R}$ is the time variable. The Toda lattice (1.5) was presented by Toda in 1967 [79]. This is an "exponential analogue" of the harmonic chain

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = (q_{n-1} - q_n) - (q_n - q_{n+1})$$

(see Figure 1.2). For the Toda lattice (1.5), one can consider several boundary conditions:

- Infinite lattice condition.
- Semi-infinite lattice condition: $q_{-1}(t) = -\infty$ for all $t \in \mathbb{R}$.
- Nonperiodic finite lattice condition: $q_{-1}(t) = -\infty$ and $q_N(t) = +\infty$ for all $t \in \mathbb{R}$, where N is a positive integer.
- Periodic finite lattice condition: $q_{N+j}(t) = q_j(t)$ for all $j \in \mathbb{Z}$ and $t \in \mathbb{R}$, where N is a positive integer.

There exist particular solutions to the Toda lattice with every boundary condition above. For details, see Toda's book [80].

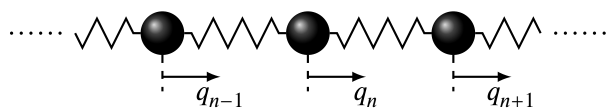


Figure 1.2: Chain of particles. If these springs obey Hooke's law, this chain is called harmonic chain.

One more example is the *Lotka–Volterra (LV) lattice* in the form

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = v_n(v_{n+1} - v_{n-1}), \quad v_n = v_n(t), \tag{1.6}$$

which is a special case of the famous prey-predator generalized LV model [16]. It is known that equation (1.6) has N-soliton solutions [30]. The LV lattice is thus also integrable.

1.1.2 Discrete integrable systems

Since there are thousands of discrete systems corresponding to a continuous system as limiting cases, *discrete integrable systems* form a wider class than continuous integrable systems. Originally, discrete integrable systems are derived by Hirota in his series of works [23–27] from continuous integrable systems through "integrable discretization", which conserves various properties of original systems attributed to their integrability.

For example, the *discrete KdV (d-KdV) lattice* [29,85] is a discrete analogue of the KdV equation (1.1):

$$u_n^{(t+1)} - u_{n-1}^{(t)} = -\delta(1+\delta)\left(\frac{1}{u_{n-1}^{(t+1)}} - \frac{1}{u_n^{(t)}}\right), \tag{1.7}$$

where $n \in \mathbb{Z}$ is the space variable, $t \in \mathbb{Z}$ is the time variable, and $\delta \in \mathbb{R}$ is a parameter. Taking a continuous limit (with complicated variable transformations), one can obtain the KdV equation (1.1) from the d-KdV lattice (1.7). One of the important properties of the d-KdV lattice (1.7) is the existence of N-soliton solutions. Through the variable transformation

$$u_n^{(t)} = (1+\delta)\frac{f_n^{(t)} f_{n+1}^{(t+1)}}{f_{n+1}^{(t)} f_n^{(t+1)}}, \tag{1.8}$$

the d-KdV lattice (1.7) is transformed into the bilinear equation

$$(1+\delta) f_n^{(t-1)} f_{n+1}^{(t+1)} = \delta f_{n+1}^{(t-1)} f_n^{(t+1)} + f_n^{(t)} f_{n+1}^{(t)}, \tag{1.9}$$

and indeed this bilinear equation has N-soliton solutions (a more generalized equation and its N-soliton solutions will be discussed in Section 4.1). Hence, the name "d-KdV lattice" is appropriate for equation (1.7).

The *discrete Toda (d-Toda) lattice* [32] is also a typical example of the discrete integrable systems:

$$q_n^{(t+1)} + e_n^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)}, \quad q_{n-1}^{(t+1)} e_n^{(t+1)} = q_n^{(t)} e_n^{(t)}. \tag{1.10}$$

Let us consider the variable transformations

$$q_n^{(t)} = J_n(t\delta) - \frac{1}{\delta}, \quad e_n^{(t)} = -\delta V_n(t\delta), \quad \delta > 0.$$

Then, we obtain

$$\begin{aligned}
\frac{J_n(t+\delta) - J_n(t)}{\delta} &= V_n(t+\delta) - V_{n+1}(t),\\
\frac{V_n(t+\delta) - V_n(t)}{\delta} &= J_{n-1}(t+\delta)V_n(t+\delta) - J_n(t)V_n(t),
\end{aligned}$$

where we replaced $t\delta$ with new t. Taking a limit $\delta \to 0$ yields

$$\frac{\mathrm{d}J_n}{\mathrm{d}t} = V_n - V_{n+1}, \quad \frac{\mathrm{d}V_n}{\mathrm{d}t} = (J_{n-1} - J_n)V_n,$$

which is Flaschka's representation of the Toda lattice [10]. Finally, consider the variable transformations

$$J_n = \frac{\mathrm{d}q_n}{\mathrm{d}t}, \quad V_n = \mathrm{e}^{q_{n-1}-q_n}.$$

Then, we arrive at the Toda lattice (1.5). Various properties of the d-Toda lattice (and its generalizations) will be discussed throughout this thesis.

The *discrete LV (d-LV) lattice* [31] is also introduced by

$$v_n^{(t+1)}(1+\delta v_{n-1}^{(t+1)}) = v_n^{(t)}(1+\delta v_{n+1}^{(t)}). \tag{1.11}$$

One can readily verify that taking a limit $\delta \to 0$ leads to the LV lattice (1.6). It is known that there exists a Miura type transformation from the d-LV lattice (1.11) to the d-Toda lattice (1.10) [71]:

$$q_n^{(t)} = (1+\delta v_{2n}^{(t)})(1+\delta v_{2n+1}^{(t)}), \quad e_n^{(t)} = v_{2n-1}^{(t)} v_{2n}^{(t)}.$$

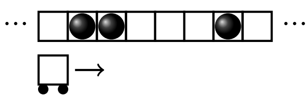
Further generalizations of this transformation will be discussed in Chapter 6.

Hirota's first work on discrete integrable systems [23] was published in 1977. The importance of discrete integrable systems is increasing as the digital computer becomes a requisite tool for any field of science and engineering. In particular, since the end of the twentieth century, a number of relationships between classical numerical algorithms and integrable systems have been studied (see the review papers [6, 8, 54]). On this basis, new algorithms based on discrete integrable systems have been developed: (i) singular value algorithms for bidiagonal matrices based on the d-LV lattice [36, 87], (ii) Padé approximation algorithms based on the discrete relativistic Toda lattice [48] and the discrete Schur flow [50], (iii) eigenvalue algorithms for band matrices based on the discrete hungry LV lattice [11] and the nonautonomous discrete hungry Toda lattice [12], and (iv) algorithms for computing D-optimal designs based on the nonautonomous d-Toda (nd-Toda) lattice [64] and the discrete mKdV (d-mKdV) lattice [65]. Applications of discrete integrable systems to numerical algorithms are considered as fascinating important topics by many researchers and engineers.

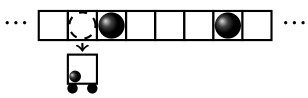
1.1.3 Ultradiscrete integrable systems

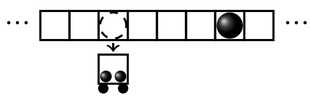
Recently, another class of integrable systems called *ultradiscrete integrable systems* has been receiving increasing attention. The study of ultradiscrete integrable systems began in 1990, when Takahashi and Satsuma [77] observed solitons in a cellular automaton, which is now called the *box–ball system* (BBS). The BBS is composed of an array of infinite boxes, finite balls in the boxes. Each box can contain only one ball. There is a carrier of balls which moves the balls from a box to another box according to the following simple rule:

- The carrier moves from left to right and passes each box once in one step. When the carrier passes a box,

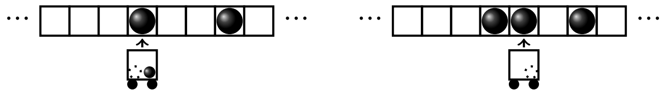


  - if there is a ball in the box, then the carrier gets it;

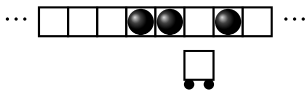




- if there is no ball in the box and the carrier holds balls, then the carrier puts a ball into the box;



- if there is no ball in the box and the carrier holds no ball, then the carrier does nothing.



Then, we can observe that the blocks of balls interacts in the same manner as solitons: a block of balls moves from left to right at constant speed depending on its size and preserves the size after interactions (see Figure 1.3). We call, therefore, each block of balls a soliton.

$\xrightarrow{n}$

```
t=0: .11111.....1111...11............................
  1: ......11111....111..111.........................
  2: ...........1111...11...11111....................
  3: ...............111..111.....11111...............
  4: ..................11...1111......11111..........
  5: ....................11.....1111.......11111.....
  6: ......................11.......1111........11111.
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Figure 1.3: Example of the time evolution of the original BBS, or the u-KdV lattice. ‘1’ denotes a ball in a box and ‘.’ denotes an empty box.

Why does the BBS possess soliton properties? In 1996, Tokihiro *et al*. [82] clarified the mathematical structure of the BBS by showing that the time evolution equation of the BBS is derived from the d-KdV lattice through a limiting procedure called *ultradiscretization*. In other words, the *ultradiscrete KdV (u-KdV) lattice* determines the time evolution of the BBS.

The fundamental formula for the ultradiscretization is given by

$$\lim_{\epsilon \to +0} -\epsilon \log(\mathrm{e}^{-A/\epsilon} + \mathrm{e}^{-B/\epsilon}) = \min(A, B), \tag{1.12}$$

where A and B are real values. The ultradiscretization is a simple transformation from the classical arithmetic operations $(+, \times)$ to the “min-plus” arithmetic operations $(\min, +)$ [17, 21]. It should be noted that the same concept is known in algebraic geometry, and called tropicalization [35].

For example, for the equation

$$a = \frac{b(c + d)}{e},$$

put $a = \mathrm{e}^{-A/\epsilon}$, $b = \mathrm{e}^{-B/\epsilon}$, $c = \mathrm{e}^{-C/\epsilon}$, $d = \mathrm{e}^{-D/\epsilon}$, $e = \mathrm{e}^{-E/\epsilon}$, and take a limit $\epsilon \to +0$. Then, we obtain the new equation

$$A = B + \min(C, D) - E.$$

That is not to say that we can ultradiscretize an arbitrary equation. Let us consider the following equation:

$$a = b - c.$$

Putting $a = \mathrm{e}^{-A/\epsilon}$, $b = \mathrm{e}^{-B/\epsilon}$, $c = \mathrm{e}^{-C/\epsilon}$, we obtain

$$A = -\epsilon \log(\mathrm{e}^{-B/\epsilon} - \mathrm{e}^{-C/\epsilon}).$$

However, the right-hand side of the above equation is indefinite if $B > C$. Therefore we cannot ultradiscretize equations containing subtractions in general. This difficulty is called "negative problem".

Let us consider the ultradiscretization of the d-KdV lattice. Since equation (1.7) includes subtraction operations, we start with its bilinear equation (1.9). Let us introduce a new dependent variable

$$z_n^{(t)} = (1+\delta)\frac{f_n^{(t)} f_{n+1}^{(t)}}{f_{n+1}^{(t-1)} f_n^{(t+1)}}.$$

Then, (1.9) yields

$$u_n^{(t+1)} = \delta(1+\delta)\frac{1}{u_n^{(t)}} + \frac{z_{n-1}^{(t+1)}}{u_{n-1}^{(t+1)}}, \tag{1.13a}$$

and the identity

$$z_n^{(t+1)} = z_{n-1}^{(t+1)} \frac{u_n^{(t)}}{u_{n-1}^{(t+1)}} \tag{1.13b}$$

holds, where $u_n^{(t)}$ is defined by (1.8). Putting $u_n^{(t)} = \mathrm{e}^{-U_n^{(t)}/\epsilon}$, $z_n^{(t)} = \mathrm{e}^{-Z_n^{(t)}/\epsilon}$, $\delta = \mathrm{e}^{-1/\epsilon}$, and taking a limit $\epsilon \to +0$, we obtain the ultradiscrete equations

$$U_n^{(t+1)} = \min\left(1 - U_n^{(t)}, Z_{n-1}^{(t+1)} - U_{n-1}^{(t+1)}\right), \tag{1.14a}$$

$$Z_n^{(t+1)} = Z_{n-1}^{(t+1)} - U_{n-1}^{(t+1)} + U_n^{(t)}, \tag{1.14b}$$

where we have used the ultradiscretization formula (1.12). We now have, from (1.14b),

$$Z_n^{(t+1)} = \sum_{j=-\infty}^{n} U_j^{(t)} - \sum_{j=-\infty}^{n-1} U_j^{(t+1)}, \tag{1.15}$$

where we impose the boundary condition $U_n^{(t)} = 0$ for $|n| \gg 1$. Substituting the expression (1.15) into (1.14a), we obtain the u-KdV lattice:

$$U_n^{(t+1)} = \min\left(1 - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)})\right). \tag{1.16}$$

Suppose that $U_n^{(t)} \in \{0, 1\}$ for all n and t. Then, the u-KdV lattice (1.16) gives the time evolution equation of the original BBS, in which $U_n^{(t)}$ denotes the number of balls in the n-th box.

Indeed, since the quantity $\sum_{j=-\infty}^{n-1}(U_j^{(t)} - U_j^{(t+1)})$ denotes the number of balls which the carrier holds at the n-th box from time t to $t+1$,

$$U_n^{(t+1)} = \begin{cases} 0 & \text{if } U_n^{(t)} = 1 \text{ or the carrier holds no ball at the } n\text{-th box,} \\ 1 & \text{if } U_n^{(t)} = 0 \text{ and the carrier holds balls at the } n\text{-th box.} \end{cases}$$

This is just the rule of the BBS already explained.

As seen above, if the initial values of the dependent variables of an ultradiscrete system are taken from integers, then the dependent variables take only integer values for all time obviously. Despite that, ultradiscrete integrable systems possess the properties attributed to the original systems: the existence of soliton solutions, infinitely many conserved quantities, and so on. Since the discovery of the ultradiscretization, ultradiscrete integrable systems have become one of the central themes in the research area of integrable systems.

Although the BBS is the first ultradiscrete integrable system which was discovered twenty years ago as mentioned above, it is being investigated actively by many researchers even now. There are several reasons for that. First, a connection between the BBS and a solvable vertex model through a limiting procedure called crystallization was discovered [19]. That is to say, we can regard the BBS as a bridge between classical integrable systems and quantum integrable systems (see also the review article [33]). Second, there are three extension rules of the BBS which we can also understand easily: box capacity [78], carrier capacity [76] and kinds of balls [81]. Third, it is known that the BBS has another time evolution equation: the *(nonperiodic) ultradiscrete finite Toda (uf-Toda) lattice* [53]. From the point of view of numerical algorithms, the last one is the main target of this thesis. The theory of BBSs may contribute effectively toward the design of numerical algorithms. The derivation of the uf-Toda lattice and its connection with the BBS will be discussed in Section 3.1.

Table 1.1 gives a summary of the difference among continuous, discrete and ultradiscrete systems.

Table 1.1: Comparison of continuous, discrete and ultradiscrete systems.

	Independent variables	Dependent variables
Continuous systems	Continuous	Continuous
Discrete systems	Discrete	Continuous
Ultradiscrete systems	Discrete	Discrete

1.2 Orthogonal polynomials and their generalizations

The concept of *orthogonality* broadly appears and is utilized in mathematics. For example, Fourier analysis provides powerful tools to decompose a function into simpler trigonometric functions. These tools have many applications to mathematics itself, physics and engineering, e.g., partial differential equations, heat transfer analysis, signal processing, and so on. The heart of Fourier analysis is the orthogonality relation of trigonometric functions. The following relation is proved easily by using the trigonometric product-to-sum formula:

$$\int_0^\pi \cos(m\theta)\cos(n\theta)\,\mathrm{d}\theta = h_n\delta_{m,n} \quad m, n = 0, 1, 2, \dots, \tag{1.17}$$

where

$$h_n := \begin{cases} \pi & \text{if } n = 0, \\ \dfrac{\pi}{2} & \text{otherwise.} \end{cases}$$

Let us now consider the polynomials defined by

$$T_n(x) := \cos(n \cos^{-1} x),$$

which means

$$T_n(\cos\theta) = \cos(n\theta).$$

From (1.17), it is shown that the polynomials $\{T_n(x)\}_{n=0}^{\infty}$ satisfy

$$\int_{-1}^{1} T_m(x)T_n(x)\frac{\mathrm{d}x}{\sqrt{1-x^2}} = h_n\delta_{m,n}.$$

The polynomials $\{T_n(x)\}_{n=0}^{\infty}$ are called the *Chebyshev polynomials of the first kind*. The addition formula of trigonometric functions leads us to the *three-term recurrence relation*:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, 3, \dots .$$

This relation implies that the degree of $T_n(x)$ is just n. Since the completeness of the Chebyshev polynomials is proved, Fourier analysis is generalized for such polynomial basis.

In general, consider a polynomial sequence $\{p_n(x)\}_{n=0}^{\infty}$ where the degree of $p_n(x)$ is just n. If these polynomials satisfy the relation

$$\int_{\Omega} p_m(x)p_n(x)w(x)\,\mathrm{d}x = h_n\delta_{m,n}, \quad h_n \neq 0, \quad m, n = 0, 1, 2, \dots,$$

where Ω is some interval on the real line and $w(x)$ is a function defined on Ω, then $\{p_n(x)\}_{n=0}^{\infty}$ are called *orthogonal polynomials*. The theory of orthogonal polynomials also have a history beginning with Stieltjes' study on continued fractions in 1894 [74], which is almost the same time as the discovery of the KdV equation in 1895. In particular, classical orthogonal polynomials, i.e. Jacobi polynomials, Laguerre polynomials and Hermite polynomials often appear in many areas of mathematical physics. The discrete version of classical orthogonal polynomials was systematically considered by Andrews and Askey [4]. These polynomials are expressed in terms of hypergeometric functions and their limit relations are drawn as the Askey-scheme [39, 40]. For further details of the theory of orthogonal polynomials, see books, e.g. Chihara [7] and Szegő [75].

Orthogonal polynomials play an important role also in the theory of integrable systems. In particular, the spectral transformation technique yields various integrable systems and particular solutions [60,71,72]. The spectral transformation for orthogonal polynomials is a mapping from an orthogonal polynomial sequence to another orthogonal polynomial sequence. We can view the three-term recurrence relation and the spectral transformation for orthogonal polynomials as a Lax pair, where the compatibility condition induces an integrable system. Furthermore, the determinant structure of orthogonal polynomials allows us to derive particular solutions to the associated integrable system. In Chapter 2, this theory will be discussed in detail.

During the last fifteen years, many researchers have extended this technique to generalized (bi)orthogonal functions and have exploited novel integrable systems that have rich properties [1–3, 38, 46, 47, 50, 51, 83, 84, 86, 89]. In this thesis, we focus on the *R_{II} polynomials* and the *R_{II} chain*. The $\mathrm{R_{II}}$ polynomials were firstly introduced by Ismail and Masson in 1995 [34] for considering generalizations of the connection between orthogonal polynomials and continued fractions. The $\mathrm{R_{II}}$ chain is a discrete integrable system derived by Spiridonov and Zhedanov in 2000 [73] as the compatibility condition of spectral transformations for the $\mathrm{R_{II}}$ polynomials. The reason why we consider the $\mathrm{R_{II}}$ chain will be given in the next section.

1.3 Outline of the thesis

The main objects of this thesis are BBSs and numerical algorithms. To discuss these subjects, we utilize discrete integrable finite lattices, i.e. discrete integrable systems with the finite lattice boundary condition, associated with (generalized) orthogonal polynomials. There are two topics:

(i) *Finite Toda representation of BBSs*. As mentioned in Section 1.1.3, the original BBS has two types of time evolution equations: the u-KdV lattice and the uf-Toda lattice. This correspondence between the u-KdV lattice and the uf-Toda lattice via the BBS is similar to the Euler–Lagrange correspondence of cellular automaton [44]. This terminology comes from hydrodynamics; the dependent variables of the Euler representation denote the number of particles at each point and the ones of the Lagrange representation denote the position of each particle. According to these definitions, we use the following terms in this thesis:

- The Euler representation of the BBS: the equation of the BBS with the variables which denote the number of balls in each box.
- The Lagrange representation of the BBS: the equation of the BBS with the variables which denote the start position of each soliton and each empty block.
- The finite Toda representation of the BBS: the equation of the BBS with the variables which denote the size of each soliton and each empty block.

The u-KdV lattice and the uf-Toda lattice denote the Euler representation and the finite Toda representation of the original BBS, respectively. Additionally, in the finite Toda representation, if we know the start position of the first soliton, we can calculate the start positions of all solitons and empty blocks from the values of the variables. In other words, the finite Toda representation and the Lagrange representation can be transformed to each other.

It has not been clarified till now why these two different ultradiscrete equations with different type boundary conditions describe the same original BBS. Moreover, the finite Toda representation of the extended BBSs are not studied sufficiently. As a first step toward these problems, one of the aims of this thesis is to derive the finite Toda representation of the BBS with variable box capacity and carrier capacity.

(ii) *New generalized eigenvalue algorithm for tridiagonal matrix pencils*. In numerical analysis, the time evolution equation of the nonautonomous df-Toda (ndf-Toda) lattice is called the *differential quotient difference with shifts (dqds) algorithm* [9]. The theory of orthogonal polynomials may allow us to extend this relationship to the $\mathrm{R_{II}}$ chain. In fact, the eigenvalue problem of tridiagonal matrices gives rise to orthogonal polynomials, and the generalized eigenvalue problem of tridiagonal matrix pencils gives rise to R_{II} polynomials [91]. Since the dqds algorithm is well known as a fast and accurate iterative algorithm for computing eigenvalues or singular values, it is worth to consider the application of the finite R_{II} chain to algorithms for computing generalized eigenvalues.

The ndf-Toda lattice appears in both topics. This means that there is an unexpected connection between BBSs and numerical algorithms. We therefore try to provide a unified framework to deal with discrete integrable finite lattices, BBSs and numerical algorithms through the theory of orthogonal polynomials. Then, this framework may offer new insights into these areas.

This thesis is organized as follows.

In Chapter 2, we recall the theory of monic orthogonal polynomials and derive the nd-Toda lattice as the compatibility condition of their spectral transformations. We discuss the properties of the ndf-Toda lattice as the quotient difference with shifts (qds) algorithm and derive its subtraction-free form, i.e. the dqds algorithm.

In Chapter 3, we first recall the correspondence between the uf-Toda lattice and the original BBS. Next, we derive the modified nonautonomous uf-Toda (nuf-Toda) lattice and discuss its connection with the BBS with a carrier; i.e. the finite Toda representation of the BBS with carrier capacity is presented. The relation between the BBS and the dqds algorithm is also discussed.

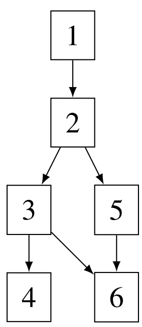
In Chapter 4, we generalize the results in Chapter 3 to the BBS with variable box capacity. First, we discuss the case of variable box capacity Δ_n and no restricted carrier capacity. After that, we discuss the case where both box capacity and carrier capacity are variable. Finally, we give a particular solution to the finite Toda representation of the BBS with fixed box capacity and a transformation from its state to a state of the BBS with box capacity 1.

In Chapter 5, we extend the theory of the dqds algorithm from the point of view of monic orthogonal polynomials in Chapter 2 to the generalized eigenvalue problem of tridiagonal matrix pencils. We prove the convergence theorem for the corresponding discrete integrable finite lattice, i.e. the monic type finite R_{II} chain, and discuss acceleration of convergence by choosing shift parameters. Furthermore, we construct a new generalized eigenvalue algorithm based on the monic type finite R_{II} chain and give numerical examples.

In Chapter 6, we first briefly recall the theory of monic symmetric orthogonal polynomials and derive a Miura type transformation between the nd-Toda lattice and the nonautonomous d-LV (nd-LV) lattice. We generalize this theory to monic symmetric R_{II} polynomials and derive a Miura type transformation between the monic type finite R_{II} chain and nonautonomous d-mKdV (nd-mKdV) lattice.

In Chapter 7, we summarize and conclude this thesis.

The following chart explains the logical dependence of the chapters:



Chapter 2

Nonautonomous Discrete Finite Toda Lattice and Eigenvalue Algorithms

Firstly, as a preliminary, we describe the essentials of the theory of orthogonal polynomials and discuss in detail the properties of the nd-Toda lattice, which is the discrete integrable lattice associated with orthogonal polynomials. We also describe how the time evolution equation of the ndf-Toda lattice is used for eigenvalue computation.

2.1 Orthogonal polynomials

Let us consider a polynomial sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ defined by the *three-term recurrence relation* of the form

$$\begin{aligned}
&\phi_{-1}(x) := 0, \quad \phi_0(x) := 1, \\
&\phi_{n+1}(x) := (x - a_n)\phi_n(x) - b_n\phi_{n-1}(x), \quad n = 0, 1, 2, \dots, \qquad (2.1)
\end{aligned}$$

where $a_n \in \mathbb{R}$ and $b_n \in \mathbb{R} - \{0\}$. By definition, $\phi_n(x)$ is a monic polynomial of degree n. Furthermore, the following classical theorem provides *orthogonality* of the polynomial sequence $\{\phi_n(x)\}_{n=0}^{\infty}$.

Theorem 2.1 (Favard's Theorem [7])**.** *For the polynomial sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ satisfying the three-term recurrence relation* (2.1) *and any constant $h_0 \in \mathbb{R} - \{0\}$, there exists a unique linear functional $\mathscr{L} : \mathbb{R}[x] \to \mathbb{R}$ such that the orthogonality relation*

$$\mathscr{L}[x^m\phi_n(x)] = h_n\delta_{m,n}, \quad n = 0, 1, 2, \dots, \quad m = 0, 1, \dots, n, \qquad (2.2)$$

holds, where

$$h_n = h_0 b_1 b_2 \dots b_n, \quad n = 1, 2, 3, \dots,$$

and $\delta_{m,n}$ is the Kronecker delta.

From the relation (2.2), we readily obtain the ordinary orthogonality relation

$$\mathscr{L}[\phi_m(x)\phi_n(x)] = h_n\delta_{m,n}, \quad m, n = 0, 1, 2, \dots. \qquad (2.3)$$

Therefore, we call the polynomials $\{\phi_n(x)\}_{n=0}^{\infty}$ the *monic orthogonal polynomials* with respect to the linear functional $\mathscr{L}$. Conversely, if a polynomial sequence $\{\phi_n(x)\}_{n=0}^{\infty}$ satisfies the ordinary orthogonality relation (2.3) with a linear functional $\mathscr{L}$, then $\{\phi_n(x)\}_{n=0}^{\infty}$ satisfies also the orthogonality relation (2.2) and the three-term recurrence relation (2.1) with some a_n and b_n.

Remark 2.2. A concrete representation of the linear functional $\mathscr{L}$ is given by a weighted integral on a real interval Ω:

$$\mathscr{L}[\pi(x)] = \int_\Omega \pi(x) w(x)\,\mathrm{d}x \quad \text{for all } \pi(x) \in \mathbb{R}[x],$$

where $w(x)$ is a weight function defined on Ω.

Remark 2.3. One may write the ordinary orthogonality relation (2.3) in terms of the inner product:

$$(\phi_m(x), \phi_n(x)) = h_n \delta_{m,n}, \quad n = 0, 1, 2, \dots, \quad m = 0, 1, \dots, n,$$

where

$$(\pi_1(x), \pi_2(x)) := \int_\Omega \pi_1(x)\pi_2(x) w(x)\,\mathrm{d}x.$$

Let us define the *moment* of order m by

$$\mu_m := \mathscr{L}[x^m], \quad m = 0, 1, 2, \dots,$$

and its *Hankel determinant* of order n by

$$\tau_{-1} := 0, \quad \tau_0 := 1, \quad \tau_n := |\mu_{i+j}|_{0 \le i,j \le n-1} = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \quad n = 1, 2, 3, \dots.$$

Since the monic orthogonal polynomial sequence $\{\phi_n(x)\}_{n=0}^\infty$ with respect to $\mathscr{L}$ is uniquely determined, we can find a determinant expression of the polynomial $\phi_n(x)$:

$$\phi_n(x) = \frac{1}{\tau_n} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_n & \mu_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ 1 & x & \dots & x^{n-1} & x^n \end{vmatrix}, \quad n = 0, 1, 2, \dots. \tag{2.4}$$

One can readily verify that the right hand side of equation (2.4) is a monic polynomial of degree n satisfying the orthogonality relation (2.2).

From the monic orthogonal polynomial sequence $\{\phi_n(x)\}_{n=0}^\infty$ with respect to $\mathscr{L}$, we can generate a new orthogonal polynomial sequence by the following procedure. Let a new linear functional $\mathscr{L}^*$ be defined by

$$\mathscr{L}^*[\pi(x)] := \mathscr{L}[(x - s)\pi(x)]$$

for all polynomials $\pi(x)$, where $s \in \mathbb{R}$ is a parameter. It follows immediately that the moments for $\mathscr{L}^*$ satisfy

$$\mu_m^* := \mathscr{L}^*[x^m] = \mu_{m+1} - s\mu_m, \quad m = 0, 1, 2, \dots.$$

We now assume that the parameter s is not a zero of $\phi_n(x)$ for all n. Then, the polynomials $\{\phi_n^*(x)\}_{n=0}^\infty$ defined by

$$\phi_n^*(x) := \frac{1}{x-s}\left(\phi_{n+1}(x) - \frac{\phi_{n+1}(s)}{\phi_n(s)}\phi_n(x)\right), \quad n = 0, 1, 2, \ldots, \tag{2.5}$$

satisfy the orthogonality relation

$$\mathscr{L}^*[x^m\phi_n^*(x)] = -\frac{\phi_{n+1}(s)}{\phi_n(s)}h_n\delta_{m,n}, \quad n = 0, 1, 2, \ldots, \quad m = 0, 1, \ldots, n.$$

Note that, by definition, the polynomial $\phi_n^*(x)$ is a monic polynomial of degree n. Hence, the polynomial sequence $\{\phi_n^*(x)\}_{n=0}^\infty$ is the monic orthogonal polynomial sequence with respect to $\mathscr{L}^*$. The procedure (2.5) is called the *Christoffel transformation* for monic orthogonal polynomials.

2.2 Derivation of the nonautonomous discrete Toda lattice and its solutions

To derive the nd-Toda lattice, we introduce discrete time $t \in \mathbb{Z}$ into the monic orthogonal polynomials as follows:

$$\phi_n^{(t+1)}(x) = \frac{\phi_{n+1}^{(t)}(x) + q_n^{(t)}\phi_n^{(t)}(x)}{x-s}, \quad q_n^{(t)} := -\frac{\phi_{n+1}^{(t)}(s^{(t)})}{\phi_n^{(t)}(s^{(t)})}, \quad n = 0, 1, 2, \ldots, \tag{2.6}$$

which means that t denotes the number of iterations of the Christoffel transformation. The reciprocal transformation called the *Geronimus transformation* is also introduced by

$$\phi_n^{(t)}(x) = \phi_n^{(t+1)}(x) + e_n^{(t)}\phi_{n-1}^{(t+1)}(x), \quad n = 0, 1, 2, \ldots, \tag{2.7}$$

where

$$e_n^{(t)} := \begin{cases} 0 & \text{if } n = 0, \\ \dfrac{\mathscr{L}^{(t)}[x^n\phi_n^{(t)}(x)]}{\mathscr{L}^{(t+1)}[x^{n-1}\phi_{n-1}^{(t+1)}(x)]} & \text{if } n = 1, 2, 3, \ldots. \end{cases}$$

Indeed, the transformations (2.6) and (2.7) restore the three-term recurrence relation

$$\phi_{n+1}^{(t)}(x) = (x - a_n^{(t)})\phi_n^{(t)}(x) - b_n^{(t)}\phi_{n-1}^{(t)}(x), \tag{2.8}$$

where $a_n^{(t)}$ and $b_n^{(t)}$ are given by

$$a_n^{(t)} = q_n^{(t)} + e_n^{(t)} + s^{(t)} = q_n^{(t-1)} + e_{n+1}^{(t-1)} + s^{(t-1)}, \tag{2.9a}$$
$$b_n^{(t)} = q_{n-1}^{(t)}e_n^{(t)} = q_n^{(t-1)}e_n^{(t-1)}. \tag{2.9b}$$

The relations (2.9) between $q_n^{(t)}$ and $e_n^{(t)}$ are the time evolution equations of the nd-Toda lattice, a nonautonomous version of the d-Toda lattice (1.10).

Remark 2.4. We can view the three-term recurrence relation (2.8) and the Geronimus transformation (2.7) rewritten in the form

$$\phi_{n+1}^{(t)}(x) + a_n^{(t)}\phi_n^{(t)}(x) + b_n^{(t)}\phi_{n-1}^{(t)}(x) = x\phi_n^{(t)}(x),$$

$$\phi_n^{(t+1)}(x) - \phi_n^{(t)}(x) = -e_n^{(t)}\phi_{n-1}^{(t+1)}(x)$$

as a discrete-time analogue of the Lax pair of the continuous-time Toda lattice [10]:

$$\begin{aligned}&\boldsymbol{\Phi}_{n+1}(t) + J_n(t)\boldsymbol{\Phi}_n(t) + V_n(t)\boldsymbol{\Phi}_{n-1}(t) = \lambda\boldsymbol{\Phi}_n(t),\\ &\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}_n(t) = -V_n(t)\boldsymbol{\Phi}_{n-1}(t).\end{aligned}$$

We have seen above the derivation of the nd-Toda lattice from the theory of monic orthogonal polynomials. Using this connection, we can give an explicit solution to the nd-Toda lattice (2.9); the solution is written in terms of the moments of the monic orthogonal polynomials. Since the moments for $\mathscr{L}^{(t)}$ satisfies the *dispersion relation*

$$\mu_m^{(t+1)} = \mu_{m+1}^{(t)} - s^{(t)}\mu_m^{(t)}, \tag{2.10}$$

we obtain

$$q_n^{(t)} = -\frac{\phi_{n+1}^{(t)}(s^{(t)})}{\phi_n^{(t)}(s^{(t)})} = \frac{\tau_n^{(t)}\tau_{n+1}^{(t+1)}}{\tau_{n+1}^{(t)}\tau_n^{(t+1)}}, \quad e_n^{(t)} = \frac{\mathscr{L}^{(t)}[x^n\phi_n^{(t)}(x)]}{\mathscr{L}^{(t+1)}[x^{n-1}\phi_{n-1}^{(t+1)}(x)]} = \frac{\tau_{n+1}^{(t)}\tau_{n-1}^{(t+1)}}{\tau_n^{(t)}\tau_n^{(t+1)}}, \tag{2.11}$$

where

$$\tau_{-1}^{(t)} = 0, \quad \tau_0^{(t)} = 1, \quad \tau_n^{(t)} = |\mu_{i+j}^{(t)}|_{0\le i,j\le n-1}, \quad n = 1,2,3,\dots, \tag{2.12}$$

and $\mu_m^{(t)}$ are arbitrary functions satisfying the dispersion relation (2.10). For example, $\mu_m^{(t)}$ are concretely given by

$$\mu_m^{(t)} = \int_\Omega x^m \prod_{j=0}^{t-1}(x - s^{(j)})w(x)\,\mathrm{d}x.$$

See also Remark 2.2.

2.3 The qds algorithm

The time evolution equations of the nd-Toda lattice (2.9) provide an algorithm for computing matrix eigenvalues or singular values. To see this, consider the following semi-infinite tridiagonal matrix:

$$\boldsymbol{B}^{(t)} = \begin{pmatrix} a_0^{(t)} & 1 & & & \\ b_1^{(t)} & a_1^{(t)} & 1 & & \\ & b_2^{(t)} & a_2^{(t)} & 1 & \\ & & b_3^{(t)} & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}.$$

Let $\boldsymbol{B}_n^{(t)}$ denote the n-th order leading principal submatrix of $\boldsymbol{B}^{(t)}$ and define the polynomials

$$\phi_0^{(t)}(x) := 1, \quad \phi_n^{(t)}(x) := \det(x\boldsymbol{I}_n - \boldsymbol{B}_n^{(t)}), \quad n = 1,2,3,\dots,$$

where $\boldsymbol{I}_n$ is the identity matrix of order n. By definition, $\phi_n^{(t)}(x)$ is a monic polynomial of degree n. Furthermore, the Laplace expansion for $\det(x\boldsymbol{I}_{n+1} - \boldsymbol{B}_{n+1}^{(t)})$ with respect to the last row yields the three-term recurrence relation (2.8). This means that the semi-infinite matrix $\boldsymbol{B}^{(t)}$ defines a monic orthogonal polynomial sequence.

Hereafter, let us consider the case of finite order; $\boldsymbol{B}^{(t)}$ is a tridiagonal matrix of order N:

$$\boldsymbol{B}^{(t)} = \begin{pmatrix} a_0^{(t)} & 1 & & & \\ b_1^{(t)} & a_1^{(t)} & 1 & & \\ & b_2^{(t)} & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b_{N-1}^{(t)} & a_{N-1}^{(t)} \end{pmatrix}, \tag{2.13}$$

where N is a positive integer. In this case, the three-term recurrence relation (2.8), which determines the finite monic orthogonal polynomial sequence $\{\phi_n^{(t)}(x)\}_{n=0}^{N}$, can be rewritten in terms of the matrix and vectors

$$\boldsymbol{B}^{(t)}\boldsymbol{\phi}^{(t)}(x) + \boldsymbol{\phi}_N^{(t)}(x) = x\boldsymbol{\phi}^{(t)}(x), \quad \boldsymbol{\phi}^{(t)}(x) := \begin{pmatrix} \phi_0^{(t)}(x) \\ \phi_1^{(t)}(x) \\ \vdots \\ \phi_{N-1}^{(t)}(x) \end{pmatrix}, \quad \boldsymbol{\phi}_N^{(t)}(x) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \phi_N^{(t)}(x) \end{pmatrix}.$$

Suppose that $x_0, x_1, \dots, x_{N-1}$ are the zeros of $\phi_N^{(t)}(x)$. Then, we obtain

$$\boldsymbol{B}^{(t)}\boldsymbol{\phi}^{(t)}(x_i) = x_i\boldsymbol{\phi}^{(t)}(x_i), \quad i = 0, 1, \dots, N-1.$$

Hence, the zeros $x_0, x_1, \dots, x_{N-1}$ of $\phi_N^{(t)}(x)$ and the vectors $\boldsymbol{\phi}^{(t)}(x_0), \boldsymbol{\phi}^{(t)}(x_1), \dots, \boldsymbol{\phi}^{(t)}(x_{N-1})$ are the eigenvalues and the eigenvectors of the tridiagonal matrix $\boldsymbol{B}^{(t)}$, respectively. Since $\phi_N^{(t)}(x) = \det(x\boldsymbol{I}_N - \boldsymbol{B}_N^{(t)}) = \det(x\boldsymbol{I}_N - \boldsymbol{B}^{(t)})$, which is the characteristic polynomial of $\boldsymbol{B}^{(t)}$, this fact is consistent.

If $\boldsymbol{B}^{(t)}$ is LU decomposable, i.e. if $\boldsymbol{B}_n^{(t)}$, $n = 1, 2, \dots, N$, are all non-singular, then $\boldsymbol{B}^{(t)}$ is decomposed into two bidiagonal matrices:

$$\boldsymbol{B}^{(t)} = \boldsymbol{L}^{(t)}\boldsymbol{R}^{(t)},$$

$$\boldsymbol{L}^{(t)} = \begin{pmatrix} 1 & & & & \\ e_1^{(t)} & 1 & & & \\ & e_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & e_{N-1}^{(t)} & 1 \end{pmatrix}, \quad \boldsymbol{R}^{(t)} = \begin{pmatrix} q_0^{(t)} & 1 & & & \\ & q_1^{(t)} & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & q_{N-1}^{(t)} \end{pmatrix}.$$

Using these bidiagonal matrices, we can rewrite the spectral transformations (2.6) and (2.7) in terms of the matrices and vectors:

$$(x - s^{(t)})\boldsymbol{\phi}^{(t+1)}(x) = \boldsymbol{R}^{(t)}\boldsymbol{\phi}^{(t)}(x) + \boldsymbol{\phi}_N^{(t)}(x), \tag{2.14a}$$

$$\boldsymbol{\phi}^{(t)}(x) = \boldsymbol{L}^{(t)}\boldsymbol{\phi}^{(t+1)}(x). \tag{2.14b}$$

Here, we impose the following condition

$$\phi_N^{(t+1)}(x) = \phi_N^{(t)}(x). \tag{2.15}$$

This condition means that the eigenvalues of $B^{(t)}$ are conserved under the time evolution and induces the additional boundary condition:

$$e_N^{(t)} = 0 \quad \text{for all } t \in \mathbb{Z}.$$

From (2.14), we obtain

$$\begin{aligned} x\boldsymbol{\phi}^{(t)}(x) - \boldsymbol{\phi}_N^{(t)}(x) &= B^{(t)}\boldsymbol{\phi}^{(t)}(x) \\ &= (L^{(t)}R^{(t)} + s^{(t)}I_N)\boldsymbol{\phi}^{(t)}(x) \\ &= (R^{(t-1)}L^{(t-1)} + s^{(t-1)}I_N)\boldsymbol{\phi}^{(t)}(x). \end{aligned}$$

Hence, as the compatibility condition for (2.14), we obtain the matrix form of the time evolution equation of the ndf-Toda lattice:

$$B^{(t)} = L^{(t)}R^{(t)} + s^{(t)}I_N = R^{(t-1)}L^{(t-1)} + s^{(t-1)}I_N. \tag{2.16}$$

This form implies that $B^{(t)}$ is similar to $B^{(t-1)}$:

$$\begin{aligned} B^{(t)} &= R^{(t-1)}L^{(t-1)} + s^{(t-1)}I_N \\ &= (L^{(t-1)})^{-1}(L^{(t-1)}R^{(t-1)} + s^{(t-1)}I_N)L^{(t-1)} \\ &= (L^{(t-1)})^{-1}B^{(t-1)}L^{(t-1)}, \end{aligned}$$

i.e. the eigenvalues of $B^{(t)}$ are conserved under the time evolution. This corresponds to the condition (2.15).

To show that we can compute the eigenvalues of the tridiagonal matrix $B^{(t)}$ by the recurrence equation (2.9), let us derive the concrete realization of the linear functional corresponding to monic orthogonal polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^N$. For the polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^N$ and any nonzero constant $h_0^{(t)}$, there exists a unique linear functional $\mathscr{L}^{(t)}$ such that the orthogonality relation

$$\mathscr{L}^{(t)}[x^m\phi_n^{(t)}(x)] = h_n^{(t)}\delta_{m,n}, \quad n = 0, 1, \dots, N-1, \quad m = 0, 1, \dots, n,$$

and the terminating condition

$$\mathscr{L}^{(t)}[x^m\phi_N^{(t)}(x)] = 0, \qquad m = 0, 1, 2, \dots,$$

hold. Let $x_0, x_1, \dots, x_{N-1}$ denote the zeros of the characteristic polynomial $\phi_N^{(t)}(x)$, i.e.

$$\phi_N^{(t)}(x) = \prod_{i=0}^{N-1}(x - x_i).$$

It is known that the value of the linear functional $\mathscr{L}^{(t)}$ is determined by the values at $x_0, x_1, \dots, x_{N-1}$. Therefore, finite orthogonal polynomials are called also *discrete orthogonal polynomials*.

Hereafter, for simplicity, we assume that the zeros $x_0, x_1, \dots, x_{N-1}$ are all simple.

Theorem 2.5 (Gauss quadrature formula [7])**.** *Let $x_0, x_1, \dots, x_{N-1}$ be the simple zeros of the characteristic polynomial $\phi_N^{(t)}(x)$. For the linear functional $\mathscr{L}^{(t)}$ of the monic finite orthogonal polynomials $\{\phi_n^{(t)}(x)\}_{n=0}^N$, there exist some nonzero constants $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$ such that*

$$\mathscr{L}^{(t)}[\pi(x)] = \sum_{i=0}^{N-1} c_i^{(t)}\pi(x_i) \tag{2.17}$$

holds for all polynomials $\pi(x)$. Furthermore, if $b_1^{(t)}, b_2^{(t)}, \dots, b_{N-1}^{(t)}$ in (2.13) *are all positive, then $c_0^{(t)}$, $c_1^{(t)}$, …, $c_{N-1}^{(t)}$ are all real and positive.*

The constants $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$ are calculated as

$$c_i^{(t)} = \frac{h_{N-1}^{(t)}}{\phi_{N-1}^{(t)}(x_i)\phi'^{(t)}_N(x_i)}, \quad i = 0, 1, \dots, N-1, \tag{2.18}$$

where $\phi'^{(t)}_N(x)$ is the derivative of $\phi_N^{(t)}(x)$. This formula is verified as follows. Due to the Gauss quadrature formula (2.17), the moment is given by

$$\mu_m^{(t)} = \mathscr{L}^{(t)}[x^m] = \sum_{i=0}^{N-1} c_i^{(t)} x_i^m, \quad m = 0, 1, 2, \dots. \tag{2.19}$$

This yields the relation

$$\mu_{m+1}^{(t)} - x_j \mu_m^{(t)} = \sum_{\substack{i=0 \\ i \neq j}}^{N-1} c_i^{(t)} (x_i - x_j) x_i^m, \quad j = 0, 1, \dots, N-1, \quad m = 0, 1, 2, \dots.$$

This relation and the determinant expression of the monic orthogonal polynomials lead to

$$\phi_{N-1}^{(t)}(x_j) = \frac{1}{\tau_{N-1}^{(t)}} \prod_{\substack{i=0 \\ i \neq j}}^{N-1} c_i^{(t)}(x_j - x_i) \prod_{\substack{0 \le \nu_0 < \nu_1 \le N-1 \\ \nu_0 \neq j, \nu_1 \neq j}} (x_{\nu_1} - x_{\nu_0})^2, \quad j = 0, 1, \dots, N-1.$$

A similar calculation yields

$$\tau_N^{(t)} = \prod_{i=0}^{N-1} c_i^{(t)} \prod_{0 \le \nu_0 < \nu_1 \le N-1} (x_{\nu_1} - x_{\nu_0})^2.$$

Furthermore, we have

$$\phi'^{(t)}_N(x_j) = \prod_{\substack{i=0 \\ i \neq j}}^{N-1} (x_j - x_i), \quad j = 0, 1, \dots, N-1,$$

$$h_{N-1}^{(t)} = \mathscr{L}^{(t)}[x^{N-1}\phi_{N-1}^{(t)}(x)] = \frac{\tau_N^{(t)}}{\tau_{N-1}^{(t)}}.$$

These equations lead to the formula (2.18).

The moments at time 0 are given by (2.19) with $t = 0$:

$$\mu_m^{(0)} = \sum_{i=0}^{N-1} c_i^{(0)} x_i^m, \quad m = 0, 1, 2, \dots.$$

Using the dispersion relation (2.10), we obtain the concrete expression of the moments at time t which consists of $c_0^{(0)}, c_1^{(0)}, \dots, c_{N-1}^{(0)}, x_0, x_1, \dots, x_{N-1}$, and the parameters $s^{(0)}, s^{(1)}, \dots, s^{(t-1)}$:

$$\mu_m^{(t)} = \sum_{i=0}^{N-1} x_i^m \eta_i^{(t)}, \quad \eta_i^{(t)} := c_i^{(0)} \prod_{j=0}^{t-1} (x_i - s^{(j)}), \quad m = 0, 1, 2, \dots.$$

Substituting this expression into (2.12) and applying the Binet–Cauchy formula [13], we obtain the expanded expression of $\tau_n^{(t)}$:

$$
\begin{aligned}
\tau_n^{(t)} &= \begin{vmatrix} \sum_{i=0}^{N-1}\eta_i^{(t)} & \sum_{i=0}^{N-1}x_i\eta_i^{(t)} & \dots & \sum_{i=0}^{N-1}x_i^{n-1}\eta_i^{(t)} \\ \sum_{i=0}^{N-1}x_i\eta_i^{(t)} & \sum_{i=0}^{N-1}x_i^2\eta_i^{(t)} & \dots & \sum_{i=0}^{N-1}x_i^{n}\eta_i^{(t)} \\ \vdots & \vdots & & \vdots \\ \sum_{i=0}^{N-1}x_i^{n-1}\eta_i^{(t)} & \sum_{i=0}^{N-1}x_i^{n}\eta_i^{(t)} & \dots & \sum_{i=0}^{N-1}x_i^{2n-2}\eta_i^{(t)} \end{vmatrix} \\
&= \det\left(\begin{pmatrix} \eta_0^{(t)} & \eta_1^{(t)} & \dots & \eta_{N-1}^{(t)} \\ x_0\eta_0^{(t)} & x_1\eta_1^{(t)} & \dots & x_{N-1}\eta_{N-1}^{(t)} \\ \vdots & \vdots & & \vdots \\ x_0^{n-1}\eta_0^{(t)} & x_1^{n-1}\eta_1^{(t)} & \dots & x_{N-1}^{n-1}\eta_{N-1}^{(t)} \end{pmatrix} \begin{pmatrix} 1 & x_0 & \dots & x_0^{n-1} \\ 1 & x_1 & \dots & x_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{N-1} & \dots & x_{N-1}^{n-1} \end{pmatrix} \right) \\
&= \sum_{0\le r_0<r_1<\dots<r_{n-1}\le N-1} \begin{vmatrix} \eta_{r_0}^{(t)} & \eta_{r_1}^{(t)} & \dots & \eta_{r_{n-1}}^{(t)} \\ x_{r_0}\eta_{r_0}^{(t)} & x_{r_1}\eta_{r_1}^{(t)} & \dots & x_{r_{n-1}}\eta_{r_{n-1}}^{(t)} \\ \vdots & \vdots & & \vdots \\ x_{r_0}^{n-1}\eta_{r_0}^{(t)} & x_{r_1}^{n-1}\eta_{r_1}^{(t)} & \dots & x_{r_{n-1}}^{n-1}\eta_{r_{n-1}}^{(t)} \end{vmatrix} \cdot \begin{vmatrix} 1 & x_{r_0} & \dots & x_{r_0}^{n-1} \\ 1 & x_{r_1} & \dots & x_{r_1}^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{r_{n-1}} & \dots & x_{r_{n-1}}^{n-1} \end{vmatrix} \\
&= \sum_{0\le r_0<r_1<\dots<r_{n-1}\le N-1} \\
&\quad \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_{r_0} & x_{r_1} & \dots & x_{r_{n-1}} \\ \vdots & \vdots & & \vdots \\ x_{r_0}^{n-1} & x_{r_1}^{n-1} & \dots & x_{r_{n-1}}^{n-1} \end{vmatrix} \cdot \begin{vmatrix} \eta_{r_0}^{(t)} & & & \\ & \eta_{r_1}^{(t)} & & \\ & & \ddots & \\ & & & \eta_{r_{n-1}}^{(t)} \end{vmatrix} \cdot \begin{vmatrix} 1 & x_{r_0} & \dots & x_{r_0}^{n-1} \\ 1 & x_{r_1} & \dots & x_{r_1}^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & x_{r_{n-1}} & \dots & x_{r_{n-1}}^{n-1} \end{vmatrix} \\
&= \sum_{0\le r_0<r_1<\dots<r_{n-1}\le N-1} \left(\prod_{i=0}^{n-1}\left(c_{r_i}^{(0)} \prod_{j=0}^{t-1}(x_{r_i}-s^{(j)}) \right) \prod_{0\le v_0<v_1\le n-1} (x_{r_{v_1}}-x_{r_{v_0}})^2 \right), \\
&\qquad\qquad n = 1,2,\dots,N. \qquad (2.20)
\end{aligned}
$$

The expanded expression of $\tau_n^{(t)}$ allows us to give the solution to the initial value problem of the ndf-Toda lattice (2.16); as same as the solution to the semi-infinite lattice (2.11), the solution to the finite lattice is given by

$$
q_n^{(t)} = \frac{\tau_n^{(t)}\tau_{n+1}^{(t+1)}}{\tau_{n+1}^{(t)}\tau_n^{(t+1)}}, \quad e_n^{(t)} = \frac{\tau_{n+1}^{(t)}\tau_{n-1}^{(t+1)}}{\tau_n^{(t)}\tau_n^{(t+1)}}, \tag{2.21}
$$

where the concrete expression of $\tau_n^{(t)}$ is (2.20).

Using the solution to the initial value problem of the ndf-Toda lattice, we show that the variables $q_n^{(t)}$ converge to the eigenvalues of the tridiagonal matrix $\boldsymbol{B}^{(0)}$ as $t \to +\infty$ under an assumption. Suppose that the subdiagonal elements $b_1^{(0)}, b_2^{(0)}, \dots, b_{N-1}^{(0)}$ of the tridiagonal matrix $\boldsymbol{B}^{(0)}$ are all positive. Then, $\boldsymbol{B}^{(0)}$ is similar to a real symmetric tridiagonal matrix and the eigenvalues $x_0, x_1, \dots, x_{N-1}$ of $\boldsymbol{B}^{(0)}$ are thus all real and simple. In addition, the constants $c_0^{(0)}, c_1^{(0)}, \dots, c_{N-1}^{(0)}$ are all real and positive by Theorem 2.5. Accordingly, the solution (2.21) with the expanded form (2.20) gives the next theorem.

Theorem 2.6. *Suppose that $b_1^{(0)}, b_2^{(0)}, \dots, b_{N-1}^{(0)}$ are all positive. Choose the parameter $s^{(t)}$ as*

$$s^{(t)} < \min\{x_0, x_1, \dots, x_{N-1}\} \quad \textit{for all } t \geq 0. \tag{2.22}$$

Then, the variables $q_n^{(t)}$ and $e_n^{(t)}$ are positive for all n and $t \geq 0$.

This theorem guarantees that zero division does not occur during the computation of the time evolution of the ndf-Toda lattice. Arrange the eigenvalues $x_0, x_1, \dots, x_{N-1}$ in descending order: $x_0 > x_1 > \cdots > x_{N-1}$. If the parameter $s^{(t)}$ is chosen as (2.22), namely $s^{(t)} < x_{N-1}$, then the inequality $x_0 - s^{(t)} > x_1 - s^{(t)} > \cdots > x_{N-1} - s^{(t)} > 0$ holds for all $t \geq 0$. Under this assumption, by the solution (2.21) with the expanded form (2.20), we obtain the asymptotic behaviour for sufficiently large t:

$$\begin{aligned} q_n^{(t)} &= x_n - s^{(t)} + O\left(\max\left\{\frac{\prod_{j=0}^{t}(x_n - s^{(j)})}{\prod_{j=0}^{t-1}(x_{n-1} - s^{(j)})}, \frac{\prod_{j=0}^{t}(x_{n+1} - s^{(j)})}{\prod_{j=0}^{t-1}(x_n - s^{(j)})}\right\}\right), \\ e_n^{(t)} &= O\left(\frac{\prod_{j=0}^{t-1}(x_n - s^{(j)})}{\prod_{j=0}^{t}(x_{n-1} - s^{(j)})}\right). \end{aligned}$$

This shows that $q_n^{(t)}$ and $e_n^{(t)}$ converge to $x_n - s^{(t)}$ and 0 as $t \to +\infty$, respectively. Hence, one can compute the eigenvalues of the tridiagonal matrix $B^{(0)}$ by Algorithm 1, which is called the qds algorithm.

Algorithm 1 The qds algorithm

---

function QDS($B^{(0)}$) $\triangleright$ $B^{(0)}$ is a tridiagonal matrix of the form (2.13)
    Set the parameter $s^{(0)}$ that satisfies the condition (2.22)
    for $n = 0, 1, \dots, N-1$ **do**
        $e_n^{(0)} := 0$ if $n = 0$, else $e_n^{(0)} := b_n^{(0)}/q_{n-1}^{(0)}$
        $q_n^{(0)} := a_n^{(0)} - e_n^{(0)} - s^{(0)}$
    end for
    $e_N^{(0)} := 0$
    $t := 0$
    repeat
        Set the parameter $s^{(t+1)}$ that satisfies the condition (2.22)
        for $n = 0, 1, \dots, N-1$ **do**
            $e_n^{(t+1)} := 0$ if $n = 0$, else $e_n^{(t+1)} := q_n^{(t)} e_n^{(t)} / q_{n-1}^{(t+1)}$
            $q_n^{(t+1)} := q_n^{(t)} - e_n^{(t+1)} + e_{n+1}^{(t)} - (s^{(t+1)} - s^{(t)})$
        end for
        $e_N^{(t+1)} := 0$
        $t := t + 1$
    until $|q_{n-1}^{(t)} e_n^{(t)}|$ are sufficiently small for all $n = 1, 2, \dots, N-1$
    return $\{q_n^{(t)} + s^{(t)}\}_{n=0}^{N-1}$
end function

It is clear that the convergence speed of the variables of the qds algorithm depends on $\frac{x_n - s^{(t)}}{x_{n-1} - s^{(t)}}$, $n = 1, 2, \dots, N-1$. Therefore, we should choose the parameter $s^{(t)} < x_{N-1}$ as close as possible to the minimum eigenvalue x_{N-1} for fast computation. The acceleration parameter $s^{(t)}$ is called *origin shift*.

2.4 The dqds algorithm

Let us write the recurrence equations of the qds algorithm again:

$$q_n^{(t+1)} = q_n^{(t)} - e_n^{(t+1)} + e_{n+1}^{(t)} - (s^{(t+1)} - s^{(t)}), \tag{2.23a}$$

$$e_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \tag{2.23b}$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t. \tag{2.23c}$$

Introduce the auxiliary variable

$$d_n^{(t+1)} := q_n^{(t+1)} - e_{n+1}^{(t)}. \tag{2.24}$$

Then, from (2.23), we obtain the relation

$$\begin{aligned}
d_n^{(t+1)} &= q_n^{(t)} - e_n^{(t+1)} - (s^{(t+1)} - s^{(t)}) \\
&= \left(q_{n-1}^{(t+1)} - \frac{q_{n-1}^{(t+1)}}{q_n^{(t)}} e_n^{(t+1)} \right) \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} - (s^{(t+1)} - s^{(t)}) \\
&= (q_{n-1}^{(t+1)} - e_n^{(t)}) \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} - (s^{(t+1)} - s^{(t)}) \\
&= d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} - (s^{(t+1)} - s^{(t)}).
\end{aligned}$$

Hence, one may use the following recurrence equations instead of (2.23):

$$d_0^{(t+1)} = q_0^{(t)} - (s^{(t+1)} - s^{(t)}), \tag{2.25a}$$

$$d_n^{(t+1)} = d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} - (s^{(t+1)} - s^{(t)}), \quad n = 1, 2, \dots, N-1, \tag{2.25b}$$

$$q_n^{(t+1)} = e_{n+1}^{(t)} + d_n^{(t+1)}, \quad n = 0, 1, \dots, N-1, \tag{2.25c}$$

$$e_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \quad n = 1, 2, \dots, N-1, \tag{2.25d}$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t. \tag{2.25e}$$

These recurrence equations are called the dqds algorithm (see also Algorithm 2).

The spectral transformations (2.6) and (2.7) yield

$$\phi_n^{(t+1)}(x) = \frac{\phi_{n+1}^{(t-1)}(x) + d_n^{(t)} \phi_n^{(t)}(x)}{x - s^{(t)}}, \quad n = 0, 1, \dots, N-1.$$

Hence, we obtain

$$d_n^{(t)} = -\frac{\phi_{n+1}^{(t-1)}(s^{(t)})}{\phi_n^{(t)}(s^{(t)})} = \frac{\tau_n^{(t)} \sigma_{n+1}^{(t)}}{\tau_{n+1}^{(t-1)} \tau_n^{(t+1)}}$$

with

$$\sigma_n^{(t)} := |\mu_{i+j+1}^{(t-1)} - s^{(t)} \mu_{i+j}^{(t-1)}|_{0 \le i,j \le n-1}.$$

Algorithm 2 The dqds algorithm

function DQDS($B^{(0)}$) $\qquad \triangleright$ $B^{(0)}$ is a tridiagonal matrix of the form (2.13)
 Set the parameter $s^{(0)}$ that satisfies the condition (2.22)
 for $n = 0, 1, \dots, N-1$ **do**
  $e_n^{(0)} := 0$ if $n = 0$, else $e_n^{(0)} := b_n^{(0)} / q_{n-1}^{(0)}$
  $q_n^{(0)} := a_n^{(0)} - e_n^{(0)} - s^{(0)}$
 end for
 $e_N^{(0)} := 0$
 $t := 0$
 repeat
  Set the parameter $s^{(t+1)}$ that satisfies the condition (2.22)
  for $n = 0, 1, \dots, N-1$ **do**
   $d_n^{(t+1)} := q_0^{(t)} - (s^{(t+1)} - s^{(t)})$ if $n = 0$, else $d_n^{(t+1)} = d_{n-1}^{(t+1)} q_n^{(t)} / q_{n-1}^{(t+1)} - (s^{(t+1)} - s^{(t)})$
   $e_n^{(t+1)} := 0$ if $n = 0$, else $e_n^{(t+1)} := q_n^{(t)} e_n^{(t)} / q_{n-1}^{(t+1)}$
   $q_n^{(t+1)} := e_{n+1}^{(t)} + d_n^{(t+1)}$
  end for
  $e_N^{(t+1)} := 0$
  $t := t + 1$
 until $|q_{n-1}^{(t)} e_n^{(t)}|$ are sufficiently small for all $n = 1, 2, \dots, N-1$
 return $\{q_n^{(t)} + s^{(t)}\}_{n=0}^{N-1}$
end function

By a calculation similar to the derivation of the expanded form (2.20), we obtain the expanded form of $\sigma_n^{(t)}$

$$\sigma_n^{(t)} = \sum_{0 \le r_0 < r_1 < \cdots < r_{n-1} \le N-1} \left(\prod_{i=0}^{n-1} \left(c_{r_i}^{(0)} (x_{r_i} - s^{(t)}) \prod_{j=0}^{t-2} (x_{r_i} - s^{(j)}) \right) \prod_{0 \le v_0 < v_1 \le n-1} (x_{r_{v_1}} - x_{r_{v_0}})^2 \right).$$

This form implies that, under the assumptions of Theorem 2.6, the variables $d_n^{(t)}$ are also positive for all n and $t \ge 0$. Therefore, the dqds algorithm enables us to compute the eigenvalues of a tridiagonal matrix without subtraction operations except the shift parameter terms $-(s^{(t)} - s^{(t-1)})$. Namely, equations (2.25) give the subtraction-free form of the time evolution equations of the ndf-Toda lattice (2.23). It is known that this form improves the accuracy of the numerical computation.

Chapter 3

Nonautonomous Ultradiscrete Finite Toda Lattice and Box–Ball System with a Carrier

In this chapter, we consider the ultradiscretization of the nd-Toda lattice, which was derived in Chapter 2. Then, we consider its connection to the BBS with a carrier [76]. This connection is an extension of the result by Nagai *et al*. [53].

3.1 Ultradiscrete finite Toda lattice and the original box–ball system

We first recall the ultradiscretization of the df-Toda lattice. The time evolution equations of the df-Toda lattice are given by

$$d_0^{(t+1)} = q_0^{(t)}, \tag{3.1a}$$

$$d_n^{(t+1)} = d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \quad n = 1, 2, \dots, N-1, \tag{3.1b}$$

$$q_n^{(t+1)} = e_{n+1}^{(t)} + d_n^{(t+1)}, \quad n = 0, 1, \dots, N-1, \tag{3.1c}$$

$$e_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \quad n = 1, 2, \dots, N-1, \tag{3.1d}$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t, \tag{3.1e}$$

which are the recurrence relations of the dqds algorithm (2.25) with $s^{(t)} = 0$ for all t (*dqd algorithm* [61]). Putting $q_n^{(t)} = \mathrm{e}^{-Q_n^{(t)}/\epsilon}$, $e_n^{(t)} = \mathrm{e}^{-E_n^{(t)}/\epsilon}$, $d_n^{(t)} = \mathrm{e}^{-D_n^{(t)}/\epsilon}$, and taking a limit $\epsilon \to +0$, we obtain the uf-Toda lattice

$$D_0^{(t+1)} = Q_0^{(t)}, \tag{3.2a}$$

$$D_n^{(t+1)} = D_{n-1}^{(t+1)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad n = 1, 2, \dots, N-1, \tag{3.2b}$$

$$Q_n^{(t+1)} = \min(E_{n+1}^{(t)}, D_n^{(t+1)}), \quad n = 0, 1, \dots, N-1, \tag{3.2c}$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad n = 1, 2, \dots, N-1, \tag{3.2d}$$

$$E_0^{(t)} = E_N^{(t)} = +\infty \quad \text{for all } t. \tag{3.2e}$$

Let the variables N, $Q_n^{(t)}$, $E_n^{(t)}$, and $D_n^{(t+1)}$ denote the following quantities of the original BBS:

- N: the number of solitons.

- $Q_n^{(t)}$: the size of the n-th soliton at time t ($n = 0, 1, \dots, N-1$);
- $E_n^{(t)}$: the size of the n-th empty block, namely, the distance between the $(n-1)$-th soliton and the n-th one at time t ($n = 1, 2, \dots, N-1$);
- $D_n^{(t+1)}$: the number of balls which the carrier holds after getting $Q_n^{(t)}$ balls ($n = 0, 1, \dots, N-1$);

Then, the next theorem gives a fundamental result on the connection between the uf-Toda lattice and the BBS.

Theorem 3.1 (Nagai *et al.* [53])**.** *The uf-Toda lattice* (3.2) *determines the time evolution of the original BBS.*

Figure 3.1 shows an example of the connection between the uf-Toda lattice and the BBS.

$\underbrace{}_{Q_0^{(0)}}\ \underbrace{}_{E_1^{(0)}}\ \underbrace{}_{Q_1^{(0)}}\ \underbrace{}_{E_2^{(0)}}\ \underbrace{}_{Q_2^{(0)}}$ $\qquad Q_0^{(t)} E_1^{(t)} Q_1^{(t)} E_2^{(t)} Q_2^{(t)}$

```
t=0: .11111.....1111...11............................. 5 5 4 3 2
  1: ......11111....111..111.......................... 5 4 3 2 3
  2: ...........1111...11...11111..................... 4 3 2 3 5
  3: ...............111..111.....11111................ 3 2 3 5 5
  4: ..................11...1111......11111........... 2 3 4 6 5
  5: ....................11.....1111.......11111...... 2 5 4 7 5
  6: ......................11.......1111........11111. 2 7 4 8 5
```

Figure 3.1: Example of the connection between the uf-Toda lattice and the original BBS. The variables $Q_n^{(t)}$ and $E_n^{(t)}$ denote the size of the n-th soliton and the one of the n-th empty block at time t, respectively.

Remark 3.2. From (3.2b) and (3.2e), the auxiliary variable $D_n^{(t+1)}$ has the following expression:

$$D_n^{(t+1)} = \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)}.$$

Thus, the uf-Toda lattice (3.2) is rewritten as

$$Q_n^{(t+1)} = \min\left(E_{n+1}^{(t)}, \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)} \right), \quad n = 0, 1, \dots, N-1, \tag{3.3a}$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \qquad n = 1, 2, \dots, N-1, \tag{3.3b}$$

$$E_0^{(t)} = E_N^{(t)} = +\infty \qquad \text{for all } t. \tag{3.3c}$$

Nagai *et al.* [53] studied not the uf-Toda lattice of the form (3.2), but of the form (3.3). However, the auxiliary variable $D_n^{(t+1)}$ plays a crucial role in the main discussion of this thesis. Therefore we do not eliminate the variable $D_n^{(t+1)}$ from the uf-Toda lattice (3.2).

Remark 3.3. Here, we remark on the Lagrange representation of the BBS. Let the variables $X_n^{(t)}$ and $Y_n^{(t)}$ denote the start position of the n-th soliton and the one of the n-th empty block at time t, respectively (see Figure 3.2). Then, the Lagrange representation of the BBS is given by [45]

$$X_n^{(t+1)} = Y_{n+1}^{(t)}, \tag{3.4a}$$

$$Y_n^{(t+1)} = Y_n^{(t)} + \min\left(X_n^{(t)} - Y_n^{(t)}, \sum_{j=1}^{n} (Y_j^{(t)} - X_{j-1}^{(t)}) - \sum_{j=1}^{n-1} (Y_j^{(t+1)} - X_{j-1}^{(t+1)}) \right), \tag{3.4b}$$

$$Y_0^{(t)} = -\infty, \quad X_N^{(t)} = +\infty. \tag{3.4c}$$

Relations between these variables and the variables of the uf-Toda lattice (3.3) are given by

$$X_n^{(t)} = Y_n^{(t)} + E_n^{(t)}, \quad Y_n^{(t)} = X_{n-1}^{(t)} + Q_{n-1}^{(t)}. \tag{3.5}$$

We can readily show that (3.4) and (3.5) yield the uf-Toda lattice (3.3). Conversely, we can calculate the values of $\{X_n^{(t)}\}_{n=1}^{N-1}$ and $\{Y_n^{(t)}\}_{n=1}^{N}$ from the values of $X_0^{(t)}$, $\{Q_n^{(t)}\}_{n=0}^{N-1}$ and $\{E_n^{(t)}\}_{n=1}^{N-1}$ using the relations (3.5). In other words, the uf-Toda lattice (3.3) and the equation $X_0^{(t+1)} = X_0^{(t)} + Q_0^{(t)}$, which is obtained from (3.4a) and (3.5), uniquely determine the time evolution of the BBS.

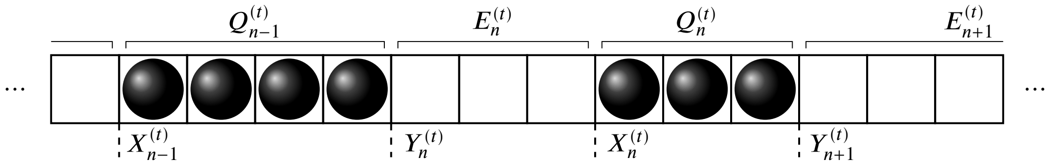


Figure 3.2: Lagrange representation and finite Toda representation of the BBS.

3.2 Ultradiscretization of the nonautonomous discrete Toda lattice

Let us consider the ultradiscretization of the ndf-Toda lattice. The time evolution equations are

$$d_0^{(t+1)} = q_0^{(t)} + \tilde{s}^{(t+1)}, \tag{3.6a}$$

$$d_n^{(t+1)} = d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} + \tilde{s}^{(t+1)}, \quad n = 1, 2, \dots, N-1, \tag{3.6b}$$

$$q_n^{(t+1)} = e_{n+1}^{(t)} + d_n^{(t+1)}, \quad n = 0, 1, \dots, N-1, \tag{3.6c}$$

$$e_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}, \quad n = 1, 2, \dots, N-1, \tag{3.6d}$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t. \tag{3.6e}$$

Now we choose the shift parameter $\tilde{s}^{(t+1)} := -(s^{(t+1)} - s^{(t)})$ as positive value. Note that, when one use the dqds algorithm, $\tilde{s}^{(t+1)}$ should not be positive value. Putting $q_n^{(t)} = \mathrm{e}^{-Q_n^{(t)}/\epsilon}$, $e_n^{(t)} = \mathrm{e}^{-E_n^{(t)}/\epsilon}$, $d_n^{(t)} = \mathrm{e}^{-D_n^{(t)}/\epsilon}$, $\tilde{s}^{(t)} = \mathrm{e}^{-\tilde{S}^{(t)}/\epsilon}$ and taking a limit $\epsilon \to +0$, we obtain the nuf-Toda lattice:

$$D_0^{(t+1)} = \min(Q_0^{(t)}, \tilde{S}^{(t+1)}), \tag{3.7a}$$

$$D_n^{(t+1)} = \min(D_{n-1}^{(t+1)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \tilde{S}^{(t+1)}), \quad n = 1, 2, \dots, N-1, \tag{3.7b}$$

$$Q_n^{(t+1)} = \min(E_{n+1}^{(t)}, D_n^{(t+1)}), \quad n = 0, 1, \dots, N-1, \tag{3.7c}$$

$$E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \quad n = 1, 2, \dots, N-1, \tag{3.7d}$$

$$E_0^{(t)} = E_N^{(t)} = +\infty \quad \text{for all } t. \tag{3.7e}$$

$Q_0^{(t)}\,E_1^{(t)}\,Q_1^{(t)}\,E_2^{(t)}\,Q_2^{(t)}$

```
t=0: .11111.....1111...11........................ 5 5 4 3 2
  1: ......11111....111..111..................... 5 4 3 2 3
  2: ...........1111...11...1111................. 4 3 2 3 4
  3: ...............111..111....1111............. 3 2 3 4 4
  4: ..................11...111.....111.......... 2 3 3 5 3
  5: ....................11....111.....111....... 2 4 3 5 3
  6: ......................11.....111.....111.... 2 5 3 5 3
  7: ........................11......111.....111. 2 6 3 5 3
```

Figure 3.3: Example of the time evolution of the nuf-Toda lattice and the corresponding BBS. The parameter $\tilde{S}^{(t)}$ is chosen as $\tilde{S}^{(t)} = +\infty$ for $t \leq 1$, $\tilde{S}^{(t)} = 4$ for $2 \leq t \leq 3$, and $\tilde{S}^{(t)} = 3$ for $t \geq 4$.

$Q_0^{(t)}\,E_1^{(t)}\,Q_1^{(t)}\,E_2^{(t)}\,Q_2^{(t)}$

```
t=0: .11111.....1111...11........................ 5 5 4 3 2
  1: ......11111....111..111..................... 5 4 3 2 3
  2: ..........11111...11...1111................. 5 3 2 3 4
  3: ..............1111..111....1111............. 4 2 3 4 4
  4: .................111..1111....1111.......... 3 2 4 4 4
  5: ....................11..11111....1111....... 2 2 5 4 4
  6: ......................11...11111....1111.... 2 3 5 4 4
  7: ........................11....11111....1111. 2 4 5 4 4
```

Figure 3.4: Example of the connection between the modified nuf-Toda lattice and the BBS with carrier capacity $S^{(t)}$. The parameter $S^{(t)}$ is chosen as $S^{(t)} = +\infty$ for $t \leq 1$, $S^{(t)} = 4$ for $2 \leq t \leq 3$, and $S^{(t)} = 3$ for $t \geq 4$.

Figure 3.3 shows an example of the time evolution of the nuf-Toda lattice and its connection to the BBS. On the other hand, Figure 3.4 shows an example of the time evolution of the BBS with a carrier, proposed by Takahashi and Matsukidaira [76]. The Euler representation of the BBS with a carrier is given by

$$U_n^{(t+1)} = \min\left(1 - U_n^{(t)},\ \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)})\right) + \max\left(0,\ \sum_{j=-\infty}^{n} U_j^{(t)} - \sum_{j=-\infty}^{n-1} U_j^{(t+1)} - S^{(t+1)}\right),$$

where the parameter $S^{(t+1)}$ is called *carrier capacity* from time t to $t+1$. The BBS corresponding to the nuf-Toda lattice shown in Figure 3.3 is very similar to the BBS with a carrier shown in Figure 3.4 except the balls in boldface. In the next section, we derive the modified version of the nuf-Toda lattice that connects to the BBS with a carrier.

3.3 Modified version of the nonautonomous ultradiscrete Toda lattice and the box–ball system with a carrier

Let us consider the monic orthogonal polynomials $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} &\phi_{-1}^{k,t}(x) := 0, \quad \phi_0^{k,t}(x) := 1, \\ &\phi_{n+1}^{k,t}(x) := (x - a_n^{k,t})\phi_n^{k,t}(x) - b_n^{k,t}\phi_{n-1}^{k,t}(x), \quad n = 0, 1, 2, \dots, \end{aligned} \tag{3.8}$$

where $a_n^{k,t} \in \mathbb{R}$, $b_n^{k,t} \in \mathbb{R} - \{0\}$, and $k, t \in \mathbb{Z}$ indicate discrete time. Let $\mathscr{L}^{k,t}$ denote a linear functional corresponding to $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$, i.e.

$$\mathscr{L}^{k,t}[x^m \phi_n^{k,t}(x)] = h_n^{k,t}\delta_{m,n}, \quad h_n^{k,t} \neq 0, \quad n = 0, 1, 2, \dots, \quad m = 0, 1, \dots, n.$$

We introduce time evolution into the monic orthogonal polynomials $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ for two directions through their spectral transformations. First, the spectral transformations for the k-direction are

$$x\phi_n^{k+1,t}(x) = \phi_{n+1}^{k,t}(x) + q_n^{k,t}\phi_n^{k,t}(x), \tag{3.9a}$$

$$\phi_n^{k,t}(x) = \phi_n^{k+1,t}(x) + e_n^{k,t}\phi_{n-1}^{k+1,t}(x), \tag{3.9b}$$

where

$$q_n^{k,t} := -\frac{\phi_{n+1}^{k,t}(0)}{\phi_n^{k,t}(0)}, \quad e_n^{k,t} := \frac{\mathscr{L}^{k,t}[x^n\phi_n^{k,t}(x)]}{\mathscr{L}^{k+1,t}[x^{n-1}\phi_{n-1}^{k+1,t}(x)]}, \tag{3.10}$$

$$\mathscr{L}^{k+1,t}[\pi(x)] := \mathscr{L}^{k,t}[x\pi(x)] \quad \text{for all } \pi(x) \in \mathbb{R}[x]. \tag{3.11}$$

Similarly, the spectral transformations for the t-direction are

$$(x + s^{(t)})\phi_n^{k,t+1}(x) = \phi_{n+1}^{k,t}(x) + \tilde{q}_n^{k,t}\phi_n^{k,t}(x), \tag{3.12a}$$

$$\phi_n^{k,t}(x) = \phi_n^{k,t+1}(x) + \tilde{e}_n^{k,t}\phi_{n-1}^{k,t+1}(x), \tag{3.12b}$$

where $s^{(t)}$ is a parameter depending on t and

$$\tilde{q}_n^{k,t} := -\frac{\phi_{n+1}^{k,t}(-s^{(t)})}{\phi_n^{k,t}(-s^{(t)})}, \quad \tilde{e}_n^{k,t} := \frac{\mathscr{L}^{k,t}[x^n\phi_n^{k,t}(x)]}{\mathscr{L}^{k,t+1}[x^{n-1}\phi_{n-1}^{k,t+1}(x)]}, \tag{3.13}$$

$$\mathscr{L}^{k,t+1}[\pi(x)] := \mathscr{L}^{k,t}[(x + s^{(t)})\pi(x)] \quad \text{for all } \pi(x) \in \mathbb{R}[x]. \tag{3.14}$$

The only difference between the transformations for the k-direction (3.9) and the t-direction (3.12) is the parameter $s^{(t)}$. As shown in Chapter 2, $\{\phi_n^{k+1,t}(x)\}_{n=0}^{\infty}$ and $\{\phi_n^{k,t+1}(x)\}_{n=0}^{\infty}$ are also the monic orthogonal polynomials with respect to $\mathscr{L}^{k+1,t}$ and $\mathscr{L}^{k,t+1}$, respectively. Figure 3.5 illustrates the relations among the monic orthogonal polynomials, the spectral transformations and the dependent variables.

Relations (3.8), (3.9) and (3.12) yield

$$\begin{aligned}
\phi_{n+1}^{k,t}(x) &= (x - a_n^{k,t})\phi_n^{k,t}(x) - b_n^{k,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (q_n^{k,t} + e_n^{k,t}))\phi_n^{k,t}(x) - q_{n-1}^{k,t}e_n^{k,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (q_n^{k-1,t} + e_{n+1}^{k-1,t}))\phi_n^{k,t}(x) - q_n^{k-1,t}e_n^{k-1,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (\tilde{q}_n^{k,t} + \tilde{e}_n^{k,t} - s^{(t)}))\phi_n^{k,t}(x) - \tilde{q}_{n-1}^{k,t}\tilde{e}_n^{k,t}\phi_{n-1}^{k,t}(x) \\
&= (x - (\tilde{q}_n^{k,t-1} + \tilde{e}_{n+1}^{k,t-1} - s^{(t-1)}))\phi_n^{k,t}(x) - \tilde{q}_n^{k,t-1}\tilde{e}_n^{k,t-1}\phi_{n-1}^{k,t}(x).
\end{aligned}$$

Hence, for consistency, the compatibility conditions

$$\begin{aligned}
a_n^{k,t} &= q_n^{k,t} + e_n^{k,t} = q_n^{k-1,t} + e_{n+1}^{k-1,t} \\
&= \tilde{q}_n^{k,t} + \tilde{e}_n^{k,t} - s^{(t)} = \tilde{q}_n^{k,t-1} + \tilde{e}_{n+1}^{k,t-1} - s^{(t-1)},
\end{aligned} \tag{3.15a}$$

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\{\phi_n^{k+1,t-1}(x)\}_{n=0}^{\infty} \arrow[r, shift left, "{\{\tilde{q}_n^{k+1,t-1}\}_{n=0}^{\infty},\ s^{(t-1)}}"] \arrow[d, shift left, "{\{e_n^{k,t-1}\}_{n=1}^{\infty}}"] & \{\phi_n^{k+1,t}(x)\}_{n=0}^{\infty} \arrow[l, shift left, "{\{\tilde{e}_n^{k+1,t-1}\}_{n=1}^{\infty}}"] \arrow[r, shift left, "{\{\tilde{q}_n^{k+1,t}\}_{n=0}^{\infty},\ s^{(t)}}"] \arrow[d, shift left, "{\{e_n^{k,t}\}_{n=1}^{\infty}}"] & \{\phi_n^{k+1,t+1}(x)\}_{n=0}^{\infty} \arrow[l, shift left, "{\{\tilde{e}_n^{k+1,t}\}_{n=1}^{\infty}}"] \arrow[d, shift left, "{\{e_n^{k,t+1}\}_{n=1}^{\infty}}"] \\
\{\phi_n^{k,t-1}(x)\}_{n=0}^{\infty} \arrow[u, shift left, "{\{q_n^{k,t-1}\}_{n=0}^{\infty}}"] \arrow[r, shift left, "{\{\tilde{q}_n^{k,t-1}\}_{n=0}^{\infty},\ s^{(t-1)}}"] \arrow[d, shift left, "{\{e_n^{k-1,t-1}\}_{n=1}^{\infty}}"] & \{\phi_n^{k,t}(x)\}_{n=0}^{\infty} \arrow[u, shift left, "{\{q_n^{k,t}\}_{n=0}^{\infty}}"] \arrow[l, shift left, "{\{\tilde{e}_n^{k,t-1}\}_{n=1}^{\infty}}"] \arrow[r, shift left, "{\{\tilde{q}_n^{k,t}\}_{n=0}^{\infty},\ s^{(t)}}"] \arrow[d, shift left, "{\{e_n^{k-1,t}\}_{n=1}^{\infty}}"] & \{\phi_n^{k,t+1}(x)\}_{n=0}^{\infty} \arrow[u, shift left, "{\{q_n^{k,t+1}\}_{n=0}^{\infty}}"] \arrow[l, shift left, "{\{\tilde{e}_n^{k,t}\}_{n=1}^{\infty}}"] \arrow[d, shift left, "{\{e_n^{k-1,t+1}\}_{n=1}^{\infty}}"] \\
\{\phi_n^{k-1,t-1}(x)\}_{n=0}^{\infty} \arrow[u, shift left, "{\{q_n^{k-1,t-1}\}_{n=0}^{\infty}}"] \arrow[r, shift left, "{\{\tilde{q}_n^{k-1,t-1}\}_{n=0}^{\infty},\ s^{(t-1)}}"] & \{\phi_n^{k-1,t}(x)\}_{n=0}^{\infty} \arrow[u, shift left, "{\{q_n^{k-1,t}\}_{n=0}^{\infty}}"] \arrow[l, shift left, "{\{\tilde{e}_n^{k-1,t-1}\}_{n=1}^{\infty}}"] \arrow[r, shift left, "{\{\tilde{q}_n^{k-1,t}\}_{n=0}^{\infty},\ s^{(t)}}"] & \{\phi_n^{k-1,t+1}(x)\}_{n=0}^{\infty} \arrow[u, shift left, "{\{q_n^{k-1,t+1}\}_{n=0}^{\infty}}"] \arrow[l, shift left, "{\{\tilde{e}_n^{k-1,t}\}_{n=1}^{\infty}}"]
\end{tikzcd}
$$

Figure 3.5: Chain of the spectral transformations for monic orthogonal polynomials.

$$
\begin{aligned}
b_n^{k,t} &= q_{n-1}^{k,t} e_n^{k,t} = q_n^{k-1,t} e_n^{k-1,t} \\
&= \tilde{q}_{n-1}^{k,t} \tilde{e}_n^{k,t} = \tilde{q}_n^{k,t-1} \tilde{e}_n^{k,t-1}, && (3.15b)\\
e_0^{k,t} &= \tilde{e}_0^{k,t} = 0 \quad \text{for all } k \text{ and } t, && (3.15c)
\end{aligned}
$$

must be satisfied. Equations (3.15) give the relations among the recurrence coefficients of the monic orthogonal polynomials $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ and the dependent variables around $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ in the diagram (Figure 3.5).

Define the moment of the linear functional $\mathscr{L}^{0,t}$ by

$$\mu_m^{(t)} := \mathscr{L}^{0,t}[x^m].$$

Note that (3.11) gives the relation

$$\mathscr{L}^{k,t}[x^m] = \mathscr{L}^{0,t}[x^{k+m}] = \mu_{k+m}^{(t)}.$$

Furthermore, (3.14) gives the dispersion relation

$$\mu_m^{(t+1)} = \mu_{m+1}^{(t)} + s^{(t)} \mu_m^{(t)}. \tag{3.16}$$

The determinant expression of the monic orthogonal polynomials $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ is given by

$$\phi_n^{k,t}(x) = \frac{1}{\tau_n^{k,t}} \begin{vmatrix} \mu_k^{(t)} & \mu_{k+1}^{(t)} & \cdots & \mu_{k+n-1}^{(t)} & \mu_{k+n}^{(t)} \\ \mu_{k+1}^{(t)} & \mu_{k+2}^{(t)} & \cdots & \mu_{k+n}^{(t)} & \mu_{k+n+1}^{(t)} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{k+n-1}^{(t)} & \mu_{k+n}^{(t)} & \cdots & \mu_{k+2n-2}^{(t)} & \mu_{k+2n-1}^{(t)} \\ 1 & x & \cdots & x^{n-1} & x^n \end{vmatrix}, \tag{3.17}$$

where $\tau_n^{k,t}$ is the Hankel determinant of order n:

$$\tau_{-1}^{k,t} := 0, \quad \tau_0^{k,t} := 1, \quad \tau_n^{k,t} := |\mu_{k+i+j}^{(t)}|_{0 \le i,j \le n-1}, \quad n = 1, 2, 3, \ldots.$$

This determinant expression (3.17) and the dispersion relation (3.16) enable us to give Hankel determinant solutions to equations (3.15); from (3.10) and (3.13), we obtain

$$q_n^{k,t} = \frac{\tau_n^{k,t}\tau_{n+1}^{k+1,t}}{\tau_{n+1}^{k,t}\tau_n^{k+1,t}}, \quad e_n^{k,t} = \frac{\tau_{n+1}^{k,t}\tau_{n-1}^{k+1,t}}{\tau_n^{k,t}\tau_n^{k+1,t}}, \tag{3.18a}$$

$$\tilde{q}_n^{k,t} = \frac{\tau_n^{k,t}\tau_{n+1}^{k,t+1}}{\tau_{n+1}^{k,t}\tau_n^{k,t+1}}, \quad \tilde{e}_n^{k,t} = \frac{\tau_{n+1}^{k,t}\tau_{n-1}^{k,t+1}}{\tau_n^{k,t}\tau_n^{k,t+1}}. \tag{3.18b}$$

We now define

$$q_n^{(t)} := q_n^{0,t+1}, \quad e_n^{(t)} := e_n^{0,t+1}, \tag{3.19a}$$

$$\tilde{q}_n^{(t)} := \tilde{q}_n^{1,t}, \quad \tilde{e}_n^{(t)} := \tilde{e}_n^{1,t}. \tag{3.19b}$$

Then, from (3.15), these variables satisfy the equations

$$\tilde{q}_n^{(t+1)} + \tilde{e}_n^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)} + s^{(t+1)}, \qquad \tilde{q}_{n-1}^{(t+1)}\tilde{e}_n^{(t+1)} = q_n^{(t)}e_n^{(t)}, \tag{3.20a}$$

$$q_n^{(t+1)} + e_{n+1}^{(t+1)} = \tilde{q}_n^{(t+1)} + \tilde{e}_{n+1}^{(t+1)} - s^{(t+1)}, \quad q_n^{(t+1)}e_n^{(t+1)} = \tilde{q}_n^{(t+1)}\tilde{e}_n^{(t+1)}. \tag{3.20b}$$

In this thesis, we call equations (3.20) the *modified nd-Toda lattice*. We consider the finite lattice condition

$$e_0^{(t)} = \tilde{e}_0^{(t)} = e_N^{(t)} = \tilde{e}_N^{(t)} = 0 \quad \text{for all } t, \tag{3.20c}$$

where N is a positive integer. Let us introduce the auxiliary variable

$$\tilde{d}_n^{(t+1)} := \tilde{q}_n^{(t+1)} - e_{n+1}^{(t)}.$$

Then, the subtraction-free form of the time evolution equations of the modified ndf-Toda lattice (3.20) is given by

$$\begin{aligned}
\tilde{d}_0^{(t+1)} &= q_n^{(t)} + s^{(t+1)}, \\
\tilde{d}_n^{(t+1)} &= \tilde{d}_{n-1}^{(t+1)}\frac{q_n^{(t)}}{\tilde{q}_{n-1}^{(t+1)}} + s^{(t+1)}, && n = 1,2,\dots,N-1, \\
\tilde{q}_n^{(t+1)} &= e_{n+1}^{(t)} + \tilde{d}_n^{(t+1)}, && n = 0,1,\dots,N-1, \\
\tilde{e}_n^{(t+1)} &= e_n^{(t)}\frac{q_n^{(t)}}{\tilde{q}_{n-1}^{(t+1)}}, && n = 1,2,\dots,N-1, \\
q_0^{(t+1)} &= \tilde{q}_0^{(t+1)}\frac{q_0^{(t)}}{\tilde{d}_0^{(t+1)}}, \\
q_n^{(t+1)} &= \tilde{q}_n^{(t+1)}\frac{\tilde{d}_{n-1}^{(t+1)}}{\tilde{d}_n^{(t+1)}}\frac{q_n^{(t)}}{\tilde{q}_{n-1}^{(t+1)}}, && n = 1,2,\dots,N-1, \\
e_n^{(t+1)} &= \tilde{e}_n^{(t+1)}\frac{\tilde{d}_n^{(t+1)}}{\tilde{d}_{n-1}^{(t+1)}}\frac{\tilde{q}_{n-1}^{(t+1)}}{q_n^{(t)}}, && n = 1,2,\dots,N-1, \\
e_0^{(t)} &= \tilde{e}_0^{(t)} = e_N^{(t)} = \tilde{e}_N^{(t)} = 0 && \text{for all } t.
\end{aligned}$$

Hence, an ultradiscrete analogue of the modified ndf-Toda lattice is given by

$$\tilde{D}_0^{(t+1)} = \min(Q_0^{(t)}, S^{(t+1)}), \tag{3.21a}$$

$$
\begin{aligned}
\tilde{D}_n^{(t+1)} &= \min(\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + Q_n^{(t)}, S^{(t+1)}), && n = 1, 2, \dots, N-1, && (3.21b)\\
\tilde{Q}_n^{(t+1)} &= \min(E_{n+1}^{(t)}, \tilde{D}_n^{(t+1)}), && n = 0, 1, \dots, N-1, && (3.21c)\\
\tilde{E}_n^{(t+1)} &= E_n^{(t)} - \tilde{Q}_{n-1}^{(t+1)} + Q_n^{(t)}, && n = 1, 2, \dots, N-1, && (3.21d)\\
Q_0^{(t+1)} &= \tilde{Q}_0^{(t+1)} + Q_0^{(t)} - \tilde{D}_0^{(t+1)}, && && (3.21e)\\
Q_n^{(t+1)} &= \tilde{Q}_n^{(t+1)} + \tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + Q_n^{(t)} - \tilde{D}_n^{(t+1)}, && n = 1, 2, \dots, N-1, && (3.21f)\\
E_n^{(t+1)} &= \tilde{E}_n^{(t+1)} - \tilde{D}_{n-1}^{(t+1)} + \tilde{Q}_{n-1}^{(t+1)} - Q_n^{(t)} + \tilde{D}_n^{(t+1)}, && n = 1, 2, \dots, N-1, && (3.21g)\\
E_0^{(t)} &= \tilde{E}_0^{(t)} = E_N^{(t)} = \tilde{E}_N^{(t)} = +\infty && \text{for all } t. && (3.21h)
\end{aligned}
$$

Equations (3.21) are composed of two maps: equations (3.21a)–(3.21d) define the map

$$\left(\{Q_n^{(t)}\}_{n=0}^{N-1}, \{E_n^{(t)}\}_{n=1}^{N-1}\right) \mapsto \left(\{\tilde{Q}_n^{(t+1)}\}_{n=0}^{N-1}, \{\tilde{E}_n^{(t+1)}\}_{n=1}^{N-1}, \{\tilde{D}_n^{(t+1)}\}_{n=0}^{N-1}\right);$$

equations (3.21e)–(3.21g) define the map

$$\left(\{\tilde{Q}_n^{(t+1)}\}_{n=0}^{N-1}, \{\tilde{E}_n^{(t+1)}\}_{n=1}^{N-1}, \{\tilde{D}_n^{(t+1)}\}_{n=0}^{N-1}\right) \mapsto \left(\{Q_n^{(t+1)}\}_{n=0}^{N-1}, \{E_n^{(t+1)}\}_{n=1}^{N-1}\right).$$

(3.21c) and (3.21d) have the same form as the uf-Toda lattice (3.2), and (3.21a) and (3.21b) describe the rule "the carrier puts $\tilde{Q}_{n-1}^{(t+1)}$ balls into boxes, gets $Q_n^{(t)}$ balls from next boxes and restricts the number of balls in the carrier to $S^{(t+1)}$ balls (excess balls vanish from the system)". Hence, (3.21a)–(3.21d) correspond to the size limit process, whose example was shown in Figure 3.3. Next, since $\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + Q_n^{(t)} - \tilde{D}_n^{(t+1)}$ gives the number of balls which are removed after the carrier gets $Q_n^{(t)}$ balls, we can view that (3.21e)–(3.21g) give the recovery process.

By summarizing the above, the next theorem is presented.

Theorem 3.4. *The modified nuf-Toda lattice* (3.21) *determines the time evolution of the BBS with carrier capacity $S^{(t+1)}$ from time t to $t+1$.*

An example of this connection was already shown in Figure 3.4.

3.4 Matrix form and particular solutions

In this section, we first investigate the matrix form of the modified ndf-Toda lattice (3.20). By using the matrix form, we give particular solutions to the modified nd-Toda lattice and discuss the asymptotic behaviour of the BBS with carrier capacity as one of the applications of the BBS for numerical algorithms.

Let us introduce bidiagonal matrices of order N as follows:

$$L^{(t)} = \begin{pmatrix} 1 & & & & \\ e_1^{(t)} & 1 & & & \\ & e_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & e_{N-1}^{(t)} & 1 \end{pmatrix}, \quad R^{(t)} = \begin{pmatrix} q_0^{(t)} & 1 & & & \\ & q_1^{(t)} & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & q_{N-1}^{(t)} \end{pmatrix},$$

$$\tilde{L}^{(t)} = \begin{pmatrix} 1 & & & & \\ \tilde{e}_1^{(t)} & 1 & & & \\ & \tilde{e}_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & \tilde{e}_{N-1}^{(t)} & 1 \end{pmatrix}, \quad \tilde{R}^{(t)} = \begin{pmatrix} \tilde{q}_0^{(t)} & 1 & & & \\ & \tilde{q}_1^{(t)} & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \tilde{q}_{N-1}^{(t)} \end{pmatrix}.$$

Then, the matrix form of the modified ndf-Toda lattice (3.20) is written as

$$\tilde{L}^{(t+1)}\tilde{R}^{(t+1)} = R^{(t)}L^{(t)} + s^{(t+1)}I, \tag{3.22a}$$

$$R^{(t+1)}L^{(t+1)} = \tilde{R}^{(t+1)}\tilde{L}^{(t+1)} - s^{(t+1)}I, \tag{3.22b}$$

where I is an identity matrix of order N.

The definition of the dependent variables (3.19) and Hankel determinant solutions (3.18) immediately yield particular solutions to the modified ndf-Toda lattice (3.20):

$$q_n^{(t)} = \frac{\tau_n^{0,t+1}\tau_{n+1}^{1,t+1}}{\tau_{n+1}^{0,t+1}\tau_n^{1,t+1}}, \quad e_n^{(t)} = \frac{\tau_{n+1}^{0,t+1}\tau_{n-1}^{1,t+1}}{\tau_n^{0,t+1}\tau_n^{1,t+1}}, \tag{3.23a}$$

$$\tilde{q}_n^{(t)} = \frac{\tau_n^{1,t}\tau_{n+1}^{1,t+1}}{\tau_{n+1}^{1,t}\tau_n^{1,t+1}}, \quad \tilde{e}_n^{(t)} = \frac{\tau_{n+1}^{1,t}\tau_{n-1}^{1,t+1}}{\tau_n^{1,t}\tau_n^{1,t+1}}. \tag{3.23b}$$

In addition, from the spectral transformations (3.9b) and (3.12a), we obtain

$$(x + s^{(t)})\phi_n^{1,t+1}(x) = \phi_{n+1}^{0,t}(x) + \tilde{d}_n^{(t)}\phi_n^{1,t}(x),$$

which implies

$$\tilde{d}_n^{(t)} = -\frac{\phi_{n+1}^{0,t}(-s^{(t)})}{\phi_n^{1,t}(-s^{(t)})} = \frac{\tau_{n+1}^{0,t+1}\tau_n^{1,t}}{\tau_{n+1}^{0,t}\tau_n^{1,t+1}}.$$

Let $x_0, x_1, \dots, x_{N-1}$ denote the eigenvalues of the tridiagonal matrix $B^{(0)} := R^{(0)}L^{(0)}$. If the variables $q_n^{(t)}$ and $e_n^{(t)}$ are all positive, then, as discussed in Chapter 2, the eigenvalues $x_0, x_1, \dots, x_{N-1}$ are all real and simple. Furthermore, suppose that $B^{(0)}$ is positive definite and arrange the eigenvalues as $x_0 > x_1 > \cdots > x_{N-1} > 0$. Moreover, there exist real and positive constants $c_0^{(0)}, c_1^{(0)}, \dots, c_{N-1}^{(0)}$ such that $\tau_n^{k,t}$ are written as

$$\tau_n^{k,t} = \sum_{0\le r_0<r_1<\cdots<r_{n-1}\le N-1}\left(\prod_{i=0}^{n-1}\left(c_{r_i}^{(0)}x_{r_i}^k\prod_{j=0}^{t-1}(x_{r_i}+s^{(j)})\right)\prod_{0\le v_0<v_1\le n-1}(x_{r_{v_1}}-x_{r_{v_0}})^2\right). \tag{3.24}$$

Choose the nonnegative parameters $s^{(t)}$. Then, in the same manner as shown in Section 2.3, the asymptotic behaviour for sufficiently large t is given by

$$q_n^{(t)} = x_n + O\left(\max\left\{\prod_{j=0}^{t}\frac{x_n+s^{(j)}}{x_{n-1}+s^{(j)}}, \prod_{j=0}^{t}\frac{x_{n+1}+s^{(j)}}{x_n+s^{(j)}}\right\}\right),$$

$$e_n^{(t)} = O\left(\prod_{j=0}^{t}\frac{x_n+s^{(j)}}{x_{n-1}+s^{(j)}}\right),$$

$$\tilde{q}_n^{(t)} = x_n + s^{(t)} + O\left(\max\left\{\frac{\prod_{j=0}^{t}(x_n+s^{(j)})}{\prod_{j=0}^{t-1}(x_{n-1}+s^{(j)})}, \frac{\prod_{j=0}^{t}(x_{n+1}+s^{(j)})}{\prod_{j=0}^{t-1}(x_n+s^{(j)})}\right\}\right),$$

$$\tilde{e}_n^{(t)} = O\left(\frac{\prod_{j=0}^{t-1}(x_n+s^{(j)})}{\prod_{j=0}^{t}(x_{n-1}+s^{(j)})}\right).$$

Next, we consider the ultradiscrete analogue of the discussion above. By ultradiscretizing (3.23) and (3.24), we obtain the following theorem.

Theorem 3.5. *A solution to the modified nuf-Toda lattice* (3.21) *is given by*

$$\begin{aligned}
Q_n^{(t)} &= T_n^{0,t+1} - T_{n+1}^{0,t+1} + T_{n+1}^{1,t+1} - T_n^{1,t+1}, \quad & E_n^{(t)} &= T_{n+1}^{0,t+1} - T_n^{0,t+1} + T_{n-1}^{1,t+1} - T_n^{1,t+1},\\
\tilde{Q}_n^{(t)} &= T_n^{1,t} - T_{n+1}^{1,t} + T_{n+1}^{1,t+1} - T_n^{1,t+1}, & \tilde{E}_n^{(t)} &= T_{n+1}^{1,t} - T_n^{1,t} + T_{n-1}^{1,t+1} - T_n^{1,t+1},\\
\tilde{D}_n^{(t)} &= T_{n+1}^{0,t+1} - T_{n+1}^{0,t} + T_n^{1,t} - T_n^{1,t+1},
\end{aligned}$$

$$T_n^{k,t} = \min_{0\le r_0<r_1<\cdots<r_{n-1}\le N-1}\left(\sum_{i=0}^{n-1}\left(C_{r_i}^{(0)} + (2(n-1-i)+k)X_{r_i} + \sum_{j=0}^{t-1}\min(X_{r_i}, S^{(j)})\right)\right), \qquad n = 1,2,\ldots,N,$$

$$T_{-1}^{k,t} = T_{N+1}^{k,t} = +\infty, \quad T_0^{k,t} = 0 \quad \textit{for all } k \textit{ and } t,$$

where X_i and $C_i^{(0)}$, $i = 0,1,\ldots,N-1$, are some constants satisfying $X_0 \le X_1 \le \ldots \le X_{N-1}$.

If, for simplicity, the parameter $S^{(t)}$ is chosen as $X_0 \le X_1 \le \ldots \le X_{m-1} < S^{(t)} \le X_m \le \ldots \le X_{N-1}$ for all t, then we have the exact expression of $T_n^{k,t}$ for sufficiently large t:

$$T_n^{k,t} = \begin{cases} 1, & n = 0,\\ \displaystyle\sum_{i=0}^{n-1}\left(C_i^{(0)} + (2(n-1-i)+k+t)X_i\right), & n = 1,2,\ldots,m-1,\\ \displaystyle\sum_{i=0}^{m-1}\left(C_i^{(0)} + (2(n-1-i)+k+t)X_i\right) + \Theta_{k,n} + (n-m)\sum_{j=0}^{t-1}S^{(j)}, & n = m, m+1,\ldots,N,\end{cases}$$

where

$$\Theta_{k,n} := \begin{cases} 0, & n = m,\\ \displaystyle\min_{m\le r_m<r_{m+1}<\cdots<r_{n-1}\le N-1}\left(\sum_{i=m}^{n-1}\left(C_{r_i}^{(0)} + (2(n-1-i)+k)X_{r_i}\right)\right), & n = m+1,\ldots,N.\end{cases}$$

We should remark that, unlike the discrete case, we can ignore all terms except the dominant term. Hence,

$$Q_n^{(t)} = \begin{cases} X_n, & n = 0,1,\ldots,m-1,\\ \Theta_{0,n} - \Theta_{0,n+1} + \Theta_{1,n+1} - \Theta_{1,n}, & n = m, m+1,\ldots,N-1,\end{cases}$$

$$E_n^{(t)} = \begin{cases} C_n^{(0)} - C_{n-1}^{(0)} + X_{n-1} + (t+1)(X_n - X_{n-1}), & n = 1,2,\ldots,m-1,\\ \Theta_{0,m+1} - C_{m-1}^{(0)} + X_{m-1} + \displaystyle\sum_{j=0}^{t} S^{(j)} - (t+1)X_{m-1}, & n = m,\\ \Theta_{0,n+1} - \Theta_{0,n} + \Theta_{1,n-1} - \Theta_{1,n}, & n = m+1, m+2,\ldots,N-1,\end{cases}$$

$$\tilde{Q}_n^{(t)} = \begin{cases} X_n, & n = 0,1,\ldots,m-1,\\ S^{(t)}, & n = m, m+1,\ldots,N-1,\end{cases}$$

$$\tilde{E}_n^{(t)} = \begin{cases} C_n^{(0)} - C_{n-1}^{(0)} + X_{n-1} + (t+1)(X_n - X_{n-1}), & n = 1, 2, \dots, m-1, \\ \Theta_{1,m+1} - C_{m-1}^{(0)} + X_{m-1} + \sum_{j=0}^{t-1} S^{(j)} - (t+1)X_{m-1}, & n = m, \\ \Theta_{1,n+1} - \Theta_{1,n} + \Theta_{1,n-1} - \Theta_{1,n} - S^{(t)}, & n = m+1, m+2, \dots, N-1, \end{cases}$$

hold for sufficiently large t. These relations indicate that, under the assumptions, $Q_n^{(t)}$ and $\tilde{Q}_n^{(t)}$ converge to some constants, and

$$E_n^{(t)} \to \begin{cases} +\infty & \text{if } n = 1, 2, \dots, m-1 \text{ and } X_{n-1} < X_n, \text{ or } n = m, \\ C_n^{(0)} - C_{n-1}^{(0)} + X_{n-1} & \text{if } n = 1, 2, \dots, m-1 \text{ and } X_{n-1} = X_n, \\ \Theta_{0,n+1} - \Theta_{0,n} + \Theta_{1,n-1} - \Theta_{1,n} & \text{if } n = m+1, m+2, \dots, N-1, \end{cases}$$

$$\tilde{E}_n^{(t)} \to \begin{cases} +\infty & \text{if } n = 1, 2, \dots, m-1 \text{ and } X_{n-1} < X_n, \text{ or } n = m, \\ C_n^{(0)} - C_{n-1}^{(0)} + X_{n-1} & \text{if } n = 1, 2, \dots, m-1 \text{ and } X_{n-1} = X_n, \\ \Theta_{1,n+1} - \Theta_{1,n} + \Theta_{1,n-1} - \Theta_{1,n} - S^{(t)} & \text{if } n = m+1, m+2, \dots, N-1, \end{cases}$$

as $t \to +\infty$, where the convergence speed depends on $\min(X_n, S^{(t)}) - \min(X_{n-1}, S^{(t)})$. In these results, we can see the correspondence between the dqds algorithm and the BBS with carrier capacity; namely, the eigenvalues in the dqds algorithm correspond to the sizes of solitons of the BBS, and the sorting property of the dqds algorithm is also observed in the time evolution of the BBS.

Chapter 4

Finite Toda Representation of Box–Ball Systems

As discussed in Chapter 3, the modified nuf-Toda lattice gives another evolution equation of the BBS with a carrier. In this chapter, we extend the above theory for more extended BBSs. In particular, we derive the finite Toda representation of the BBS with variable box capacity and carrier capacity.

4.1 Euler representation of the box–ball system with variable box capacity and carrier capacity

First, we give the Euler representation of the generalized BBS. The nonautonomous discrete KP (nd-KP) equation is given by [90]

$$(a_n - b_t) f_{n+1}^{k,t+1} f_n^{k+1,t} + (b_t - c_k) f_{n+1}^{k,t} f_n^{k+1,t+1} + (c_k - a_n) f_n^{k,t+1} f_{n+1}^{k+1,t} = 0, \quad k, n, t \in \mathbb{Z}. \tag{4.1}$$

It is shown that an N-soliton solution to the nd-KP equation (4.1) is presented by

$$f_n^{k,t} = 1 + \sum_{\substack{J \subset \{0,1,\ldots,N-1\} \\ J \neq \emptyset}} \left(\prod_{\substack{i,j \in J \\ i \neq j}} w_{i,j} \prod_{i \in J} h_{i,n}^{k,t} \right),$$

$$h_{i,n}^{k,t} := \xi_i \prod_{j=0}^{n-1} \frac{a_j - x_i}{a_j - y_i} \prod_{j=0}^{t-1} \frac{b_j - x_i}{b_j - y_i} \prod_{j=0}^{k-1} \frac{c_j - x_i}{c_j - y_i}, \quad w_{i,j} := \frac{(x_i - x_j)(y_i - y_j)}{(x_i - y_j)(y_i - x_j)},$$

where ξ_i, x_i and y_i, $i = 0, 1, \ldots, N-1$, are some constants. Now we impose the 2-reduction condition with respect to the variable k, that is $f_n^{k+2,t} = f_n^{k,t}$ and $c_{k+2} = c_k$ for all $k \in \mathbb{Z}$, and set $a_n = 1 + \delta_n$, $b_t = -s^{(t)}$, $c_0 = 1$ and $c_1 = 0$. Then, the nd-KP equation (4.1) reduces to the forms

$$(1 + \delta_n + s^{(t)}) f_{n+1}^{0,t+1} f_n^{1,t} = (1 + s^{(t)}) f_{n+1}^{0,t} f_n^{1,t+1} + \delta_n f_n^{0,t+1} f_{n+1}^{1,t}, \tag{4.2a}$$

$$(1 + \delta_n + s^{(t)}) f_n^{0,t} f_{n+1}^{1,t+1} = (1 + \delta_n) f_{n+1}^{0,t} f_n^{1,t+1} + s^{(t)} f_n^{0,t+1} f_{n+1}^{1,t}, \tag{4.2b}$$

and an N-soliton solution to the reduced equations is given by

$$f_n^{k,t} = 1 + \sum_{\substack{J \subset \{0,1,\ldots,N-1\} \\ J \neq \emptyset}} \left(\prod_{\substack{i,j \in J \\ i \neq j}} w_{i,j} \prod_{i \in J} h_{i,n}^{k,t} \right), \quad k = 0, 1, \tag{4.3a}$$

$$h_{i,n}^{0,t} \coloneqq \xi_i \prod_{j=0}^{n-1} \frac{1+\delta_j - x_i}{x_i + \delta_j} \prod_{j=0}^{t-1} \frac{x_i + s^{(j)}}{1 + s^{(j)} - x_i}, \quad h_{i,n}^{1,t} \coloneqq \frac{1-x_i}{x_i} h_{i,n}^{0,t}, \tag{4.3b}$$

$$w_{i,j} \coloneqq \left(\frac{x_i - x_j}{1 - x_i - x_j} \right)^2. \tag{4.3c}$$

Let us define the dependent variables as

$$u_n^{(t)} = \frac{f_{n+1}^{0,t+1} f_n^{1,t+1}}{f_n^{0,t+1} f_{n+1}^{1,t+1}}, \quad \tilde{u}_n^{(t)} = (1 + \delta_n + s^{(t)}) \frac{f_n^{0,t} f_{n+1}^{0,t+1}}{f_{n+1}^{0,t} f_n^{0,t+1}}, \quad \tilde{z}_n^{(t)} = \frac{f_n^{0,t} f_n^{1,t+1}}{f_n^{0,t+1} f_n^{1,t}}. \tag{4.4}$$

Then, the 2-reduced nd-KP equation (4.2) yields the equations

$$\tilde{u}_n^{(t+1)} = \delta_n \frac{1}{u_n^{(t)}} + (1 + s^{(t+1)}) \tilde{z}_n^{(t+1)}, \tag{4.5a}$$

$$\tilde{z}_n^{(t+1)} = \frac{(1+\delta_n) \tilde{z}_{n-1}^{(t+1)} u_{n-1}^{(t)} + s^{(t+1)}}{\tilde{u}_{n-1}^{(t+1)}}, \tag{4.5b}$$

and the identity

$$u_n^{(t+1)} = u_n^{(t)} \frac{\tilde{z}_n^{(t+1)}}{\tilde{z}_{n+1}^{(t+1)}} \tag{4.5c}$$

holds. For positivity, we choose the parameters as $0 \le \delta_n \le 1$ and $0 \le s^{(t)} \le 1$ for all $n, t \in \mathbb{Z}$. If the values of the dependent variables are all positive for all $n, t \in \mathbb{Z}$, we can ultradiscretize equations (4.5): putting $u_n^{(t)} = \mathrm{e}^{-U_n^{(t)}/\epsilon}$, $\tilde{u}_n^{(t)} = \mathrm{e}^{-\tilde{U}_n^{(t)}/\epsilon}$, $\tilde{z}_n^{(t)} = \mathrm{e}^{-\tilde{Z}_n^{(t)}/\epsilon}$, $\delta_n = \mathrm{e}^{-\Delta_n/\epsilon}$, $s^{(t)} = \mathrm{e}^{-S^{(t)}/\epsilon}$ into (4.5) and taking a limit $\epsilon \to +0$, we obtain the 2-reduced nonautonomous ultradiscrete KP (nu-KP) equation

$$\tilde{U}_n^{(t+1)} = \min(\Delta_n - U_n^{(t)}, \tilde{Z}_n^{(t+1)}), \tag{4.6a}$$

$$\tilde{Z}_n^{(t+1)} = \min(\tilde{Z}_{n-1}^{(t+1)} + U_{n-1}^{(t)}, S^{(t+1)}) - \tilde{U}_{n-1}^{(t+1)}, \tag{4.6b}$$

$$U_n^{(t+1)} = U_n^{(t)} + \tilde{Z}_n^{(t+1)} - \tilde{Z}_{n+1}^{(t+1)}, \tag{4.6c}$$

where $\Delta_n, S^{(t)} \ge 0$ for all $n, t \in \mathbb{Z}$. An N-soliton solution to the ultradiscrete system (4.6) is obtained as follows. Let us take the constants x_i, $i = 0, 1, \dots, N-1$, to satisfy the condition $0 < x_i < 1$. Putting $f_n^{k,t} = \mathrm{e}^{-F_n^{k,t}/\epsilon}$, $h_{i,n}^{k,t} = \mathrm{e}^{-H_{i,n}^{k,t}/\epsilon}$, $x_i = \mathrm{e}^{-X_i/\epsilon}$, $\xi_i = \mathrm{e}^{-\Xi_i/\epsilon}$, $w_{i,j} = \mathrm{e}^{-W_{i,j}/\epsilon}$ into (4.3) and (4.4), and taking a limit $\epsilon \to +0$, we obtain

$$U_n^{(t)} = F_{n+1}^{0,t+1} - F_n^{0,t+1} + F_n^{1,t+1} - F_{n+1}^{1,t+1},$$
$$\tilde{U}_n^{(t)} = F_n^{0,t} - F_{n+1}^{0,t} + F_{n+1}^{0,t+1} - F_n^{0,t+1},$$
$$\tilde{Z}_n^{(t)} = F_n^{0,t} - F_n^{0,t+1} + F_n^{1,t+1} - F_n^{1,t},$$
$$F_n^{k,t} = \min\left(0, \min_{\substack{J \subset \{0,1,\dots,N-1\} \\ J \ne \emptyset}} \left(\sum_{\substack{i,j \in J \\ i \ne j}} W_{i,j} + \sum_{i \in J} H_{i,n}^{k,t} \right) \right), \quad k = 0, 1,$$
$$H_{i,n}^{0,t} = \Xi_i - \sum_{j=0}^{n-1} \min(X_i, \Delta_j) + \sum_{j=0}^{t-1} \min(X_i, S^{(j)}), \quad H_{i,n}^{1,t} = H_{i,n}^{0,t} - X_i,$$

$$W_{i,j} = 2\min(X_i, X_j),$$

and $X_i \geq 0$, $i = 0, 1, \dots, N-1$.

Let us introduce the time evolution rule of the BBS with the n-th box capacity Δ_n and the carrier capacity $S^{(t+1)}$ from time t to $t+1$. We consider the time evolution rule from time t to $t+1$ as the composition of *size limit process* and *recovery process*.

(i) Size limit process: the carrier of balls moves from left ($n = -\infty$) to right ($n = +\infty$). When the carrier passes each box, the carrier gets all balls in the box; and if the number of balls exceeds the carrier capacity $S^{(t+1)}$, the excess balls are removed from the system. At the same time, the carrier puts the balls in the carrier into the box as many as possible.

(ii) Recovery process: after the size limit process, all the removed balls are recovered to the boxes in which the balls were.

Figure 4.1 illustrates these rules.

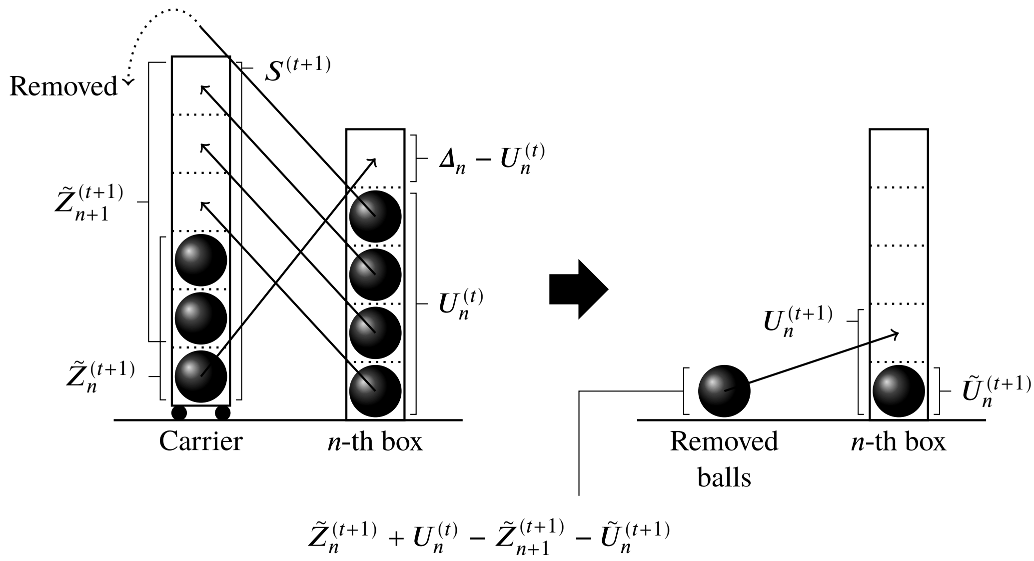


Figure 4.1: Illustration of the time evolution rule of the BBS with a carrier. The left figure illustrates the size limit process and the right one illustrates the recovery process.

Suppose that the dependent variables $U_n^{(t)}$, $\tilde{U}_n^{(t+1)}$ and $\tilde{Z}_n^{(t+1)}$ denote the following quantities:

- $U_n^{(t)} \in \{0, 1, \dots, \Delta_n\}$: the number of balls in the n-th box at time t;
- $\tilde{U}_n^{(t+1)} \in \{0, 1, \dots, \Delta_n\}$: the number of balls in the n-th box after the size limit process from time t to $t+1$;
- $\tilde{Z}_n^{(t+1)} \in \{0, 1, \dots, S^{(t+1)}\}$: the number of balls which the carrier holds arriving at the n-th box in the size limit process from time t to $t+1$.

Then, equations (4.6) give the time evolution rule: equations (4.6a) and (4.6b) define the size limit process

$$\{U_n^{(t)}\}_{n=-\infty}^{+\infty} \mapsto \left(\{\tilde{U}_n^{(t+1)}\}_{n=-\infty}^{+\infty}, \{\tilde{Z}_n^{(t+1)}\}_{n=-\infty}^{+\infty}\right)$$

and, since

$$\begin{aligned}
&\tilde{U}_n^{(t+1)} + \left((\tilde{Z}_n^{(t+1)} + U_n^{(t)}) - \min(\tilde{Z}_n^{(t+1)} + U_n^{(t)}, S^{(t+1)})\right) \\
&= \tilde{U}_n^{(t+1)} + \tilde{Z}_n^{(t+1)} + U_n^{(t)} - \tilde{Z}_{n+1}^{(t+1)} - \tilde{U}_n^{(t+1)} \\
&= U_n^{(t)} + \tilde{Z}_n^{(t+1)} - \tilde{Z}_{n+1}^{(t+1)}
\end{aligned}$$

gives the number of removed balls by the size limit at the n-th box, equation (4.6c) defines the recovery process

$$\left(\{\tilde{U}_n^{(t+1)}\}_{n=-\infty}^{+\infty}, \{\tilde{Z}_n^{(t+1)}\}_{n=-\infty}^{+\infty}\right) \mapsto \{U_n^{(t+1)}\}_{n=-\infty}^{+\infty}.$$

	$\tilde{U}_n^{(t)}$:		$U_n^{(t)}$:
Δ_n:	35353535353535353535	Δ_n:	35353535353535353535
t=0:	15213..2............	t=0:	15213..2............
1:	..14.5..2...........	1:	..**3**4.5..2...........
2:	...13.33.2..........	2:	...**231**33.2..........
3:	.....4.2312.........	3:	.....4.**4**312.........
4:	......31.413........	4:	......31**15**13........
5:	.......31.2231......	5:	.......31.**33**31......
6:	........22.2.42.....	6:	........22.2**15**2.....
7:	.........22.2.15....	7:	.........22.2.**3**5....
8:	..........13.2..33..	8:	..........13.2.**2**33..
9:	...........13.2..231	9:	...........13.2..**4**31

Figure 4.2: Example of the 3-soliton solution to the 2-reduced nu-KP equation. The leftmost box is the 0th box in the both figures. The carrier capacity $S^{(t)} = 6$ for all $t \geq 1$. Each number denotes the number of balls in a box and '.' denotes an empty box. In the right figure, boxes containing recovered balls are shown in boldface (compare to the left figure).

Figure 4.2 shows an example of the 3-soliton solution to the time evolution equation (4.6). The parameters are chosen as

$$\Delta_n = \begin{cases} 3 & \text{if } n \text{ is even,} \\ 5 & \text{if } n \text{ is odd,} \end{cases} \qquad S^{(t)} = \begin{cases} +\infty & \text{if } t \leq 0, \\ 6 & \text{if } t > 0. \end{cases}$$

Remark 4.1. Eliminating the variable $\tilde{U}_n^{(t+1)}$ from equations (4.6a) and (4.6b), we have the relation

$$-\tilde{Z}_{n+1}^{(t+1)} = \min(\Delta_n - U_n^{(t)}, \tilde{Z}_n^{(t+1)}) - \min(\tilde{Z}_n^{(t+1)} + U_n^{(t)}, S^{(t+1)}). \tag{4.7}$$

From equation (4.6c), the relation

$$\tilde{Z}_n^{(t+1)} = \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)}) \tag{4.8}$$

also holds. Substituting (4.8) into (4.7), we obtain the equation

$$U_n^{(t+1)} = \min\left(\Delta_n - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)})\right) + \max\left(0, \sum_{j=-\infty}^{n} U_j^{(t)} - \sum_{j=-\infty}^{n-1} U_j^{(t+1)} - S^{(t+1)}\right), \quad (4.9)$$

where we have used the formula

$$-\min(-A, -B) = \max(A, B). \quad (4.10)$$

Equation (4.9) has the same form as of the time evolution equation of the BBS with a carrier.

If we choose $S^{(t)} = +\infty$ for all t, then (4.9) yields

$$U_n^{(t+1)} = \min\left(\Delta_n - U_n^{(t)}, \sum_{j=-\infty}^{n-1} (U_j^{(t)} - U_j^{(t+1)})\right), \quad (4.11)$$

which is the nonautonomous ultradiscrete KdV (nu-KdV) lattice. In addition, (4.6b) yields

$$\tilde{Z}_n^{(t+1)} = \tilde{Z}_{n-1}^{(t+1)} + U_{n-1}^{(t)} - \tilde{U}_{n-1}^{(t+1)},$$

and comparing this relation with (4.6c), we have the relation $U_n^{(t+1)} = \tilde{U}_n^{(t+1)}$.

4.2 Finite Toda representation of the box–ball system with variable box capacity

In previous studies, the finite Toda representation is considered only for the BBS with box capacity 1. In this section, we extend the finite Toda representation to the case in which each box has own capacity Δ_n. First, we consider the case of carrier capacity $S^{(t)} = +\infty$. The Euler representation of this case is given by (4.11).

We first define the size of solitons and the one of empty blocks for the BBS with variable box capacity Δ_n at any time t. For this purpose, we refer to the work by Takahashi and Satsuma [78]. They analyzed the BBS with the fixed box capacity Δ using a map from a state of box capacity Δ to a binary sequence. We generalize this map for the case of variable box capacity Δ_n.

Suppose that a state of the Euler representation $\{U_n^{(t)}\}_{n=-\infty}^{+\infty}$ such that $U_n^{(t)} \in \{0, 1, \dots, \Delta_n\}$ is given. We assume that, for simplicity, $U_n^{(t)} = 0$ for $n < 0$. Let us define a map $\{U_n^{(t)}\}_{n=-\infty}^{+\infty} \mapsto \{V_n^{(t)}\}_{n=-\infty}^{+\infty}$, where $V_n^{(t)} \in \{0, 1\}$, as follows:

(1) $V_n^{(t)} = 0$ for $n < 0$.

(2) Let $i_0 = 0$ and $i_n = \sum_{j=0}^{n-1} \Delta_j$ for $n = 1, 2, \dots$. From $n = 0$ to $+\infty$, if $V_{i_n-1}^{(t)} = 1$, then

$$V_{i_n}^{(t)} = V_{i_n+1}^{(t)} = \cdots = V_{i_n+U_n^{(t)}-1}^{(t)} = 1,$$
$$V_{i_n+U_n^{(t)}}^{(t)} = V_{i_n+U_n^{(t)}+1}^{(t)} = \cdots = V_{i_n+\Delta_n-1}^{(t)} = 0;$$

otherwise,

$$V_{i_n}^{(t)} = V_{i_n+1}^{(t)} = \cdots = V_{i_n-U_n^{(t)}-1+\Delta_n}^{(t)} = 0,$$
$$V_{i_n-U_n^{(t)}+\Delta_n}^{(t)} = V_{i_n-U_n^{(t)}+\Delta_n+1}^{(t)} = \cdots = V_{i_n+\Delta_n-1}^{(t)} = 1.$$

Note that the relation $U_n^{(t)} = \sum_{j=i_n}^{i_n+\Delta_n-1} V_j^{(t)}$ holds.

We refer the j-th number $V_j^{(t)}$ in the binary sequence as the j-th *segment*. By using this map, the n-th box is expanded to the block composed from the i_n-th to $(i_n + \varDelta_n - 1)$-th segments. We call this map *expansion map* from a state of the BBS with variable box capacity $\varDelta_n$ to a binary sequence.

```
          Δn:  5353535353
       Un(t):  .15213..2.
                   ↧
                                                        Q0(t) E1(t) Q1(t) E2(t) Q2(t)
Vj(t): |.....|..1|11111|11.|....1|111|.....|...|...11|...|  8  5  4  11  2
```

Figure 4.3: Example of the expansion map from a state of the BBS with variable box capacity $\varDelta_n$ to a binary sequence. In the binary sequence, a block between two separators '|' corresponds to an original box.

```
  Δn: 3535353535353535
 ---------------------
t=0: 15213..2........
  1: ..14.52.2.......
  2: ...13.1513......
  3: .....4..2233....
  4: ......31.2.233..
  5: .......31.2..233
                 ↧
                                                                                   Q0(t) E1(t) Q1(t) E2(t) Q2(t)
t=0: |..1|11111|11.|....1|111|.....|...|...11|...|.....|...|.....|...|.....|...|.....|  8   5  4  11  2
  1: |...|.....|..1|1111.|...|11111|11.|.....|.11|.....|...|.....|...|.....|...|.....|  5   4  7   7  2
  2: |...|.....|...|....1|111|.....|..1|11111|1..|..111|...|.....|...|.....|...|.....|  4   7  7   4  3
  3: |...|.....|...|.....|...|.1111|...|.....|.11|11...|111|111..|...|.....|...|.....|  4   9  4   3  6
  4: |...|.....|...|.....|...|.....|111|1....|...|...11|...|...11|111|111..|...|.....|  4  10  2   6  8
  5: |...|.....|...|.....|...|.....|...|..111|1..|.....|.11|.....|...|...11|111|111..|  4   8  2  11  8
```

Figure 4.4: Example of the expansion map for the BBS with variable box capacity $\varDelta_n$. The carrier capacity parameter is chosen as $S^{(t)} = +\infty$ for all $t \in \mathbb{Z}$.

Figures 4.3 and 4.4 show examples of the expansion map. As shown in Figure 4.4, the expansion map enables us to define the size of the n-th soliton $Q_n^{(t)}$ and the one of the n-th empty block $E_n^{(t)}$ for the BBS with box capacity $\varDelta_n$ at any time t in the same way as for the BBS with box capacity 1. Let $D_n^{(t+1)}$ denote the number of balls which the carrier holds after getting $Q_n^{(t)}$ balls, and $\varLambda_n^{(t)}$ denote the capacity of the box which contains the beginning (leftmost) segment of the n-th empty block. Then, we arrive at the following theorem.

Theorem 4.2. *Let the variables $Q_n^{(t)}$, $E_n^{(t)}$ and $D_n^{(t+1)}$ denote the quantities of the BBS as explained in Section 3.1. Then, the time evolution of the BBS with variable box capacity $\varLambda_n^{(t)}$ is given by*

$$D_0^{(t+1)} = Q_0^{(t)}, \tag{4.12a}$$

$$D_n^{(t+1)} = D_{n-1}^{(t+1)} - Q_{n-1}^{(t+1)} + Q_n^{(t)}, \tag{4.12b}$$

$$Q_n^{(t+1)} = \min\left(E_{n+1}^{(t)} - \max(0, \varLambda_{n+1}^{(t)} - D_n^{(t+1)}), D_n^{(t+1)}\right), \tag{4.12c}$$

$$\begin{aligned} E_n^{(t+1)} = E_n^{(t)} - Q_{n-1}^{(t+1)} + Q_n^{(t)} \\ - \max(0, \varLambda_n^{(t)} - D_{n-1}^{(t+1)}) + \max(0, \varLambda_{n+1}^{(t)} - D_n^{(t+1)}), \end{aligned} \tag{4.12d}$$

$$E_0^{(t)} = E_N^{(t)} = +\infty \quad \textit{for all } t. \tag{4.12e}$$

We note that, from (4.12c), $Q_n^{(t+1)} \leq D_n^{(t+1)}$ holds for all n and t. Since the size of the n-th soliton $Q_n^{(t)}$ should be positive for all n and t, from (4.12b), the inequality $D_n^{(t+1)} \geq 1$ holds for all n and t. Thus, all the terms $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ are equal to zero when $\Lambda_n^{(t)} = 1$ for all n and t, the case of the original BBS. In this case, equations (4.12) reduce to the finite Toda representation of the original BBS (3.2). Hence, we can say that the ultradiscrete system (4.12) is a generalization of the uf-Toda lattice (3.2).

Proof. Let us consider the general $\Lambda_n^{(t)} \geq 1$ case. As we mentioned above, (4.12) has additional terms $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ which do not appear in the case of box capacity 1 (3.2). Hence, we shall investigate the role of the terms $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$.

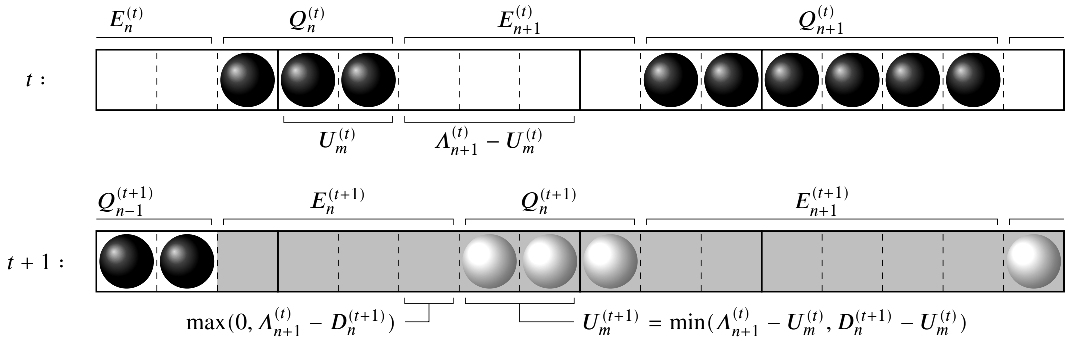


Figure 4.5: Illustration of the quantity $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$. We now need to determine the quantities on the area filled with gray: $E_n^{(t+1)}, Q_n^{(t+1)}, E_{n+1}^{(t+1)}, \ldots, Q_{N-1}^{(t+1)}$. In this figure, $D_{n-1}^{(t+1)} - Q_{n-1}^{(t+1)} = 1$ is assumed; the carrier has one ball just before getting $Q_n^{(t)}$ balls.

Let us consider the time evolution of the BBS with box capacity Δ_n from time t to $t+1$. Assume that $Q_j^{(t+1)}$, $j = 0, 1, \ldots, n-1$, and $E_j^{(t+1)}$, $j = 1, 2, \ldots, n-1$, are given (see Figure 4.5). Let m be the index of the box which contains the leftmost segment of the $(n+1)$th empty block at time t. Then, the capacity of the m-th box Δ_m is equal to $\Lambda_{n+1}^{(t)}$ by definition. Moreover, the relation

$$D_n^{(t+1)} = \sum_{j=0}^{n} Q_j^{(t)} - \sum_{j=0}^{n-1} Q_j^{(t+1)} = \sum_{j=-\infty}^{m-1} (U_j^{(t)} - U_j^{(t+1)}) + U_m^{(t)},$$

where $U_k^{(t)}$ denotes the number of balls in the k-th box at time t, also holds by definition. Hence, we can calculate the quantity $U_m^{(t+1)}$ by the nu-KdV lattice (4.11):

$$\begin{aligned} U_m^{(t+1)} &= \min\left(\Delta_m - U_m^{(t)}, \sum_{j=-\infty}^{m-1} (U_j^{(t)} - U_j^{(t+1)})\right) \\ &= \min(\Lambda_{n+1}^{(t)} - U_m^{(t)}, D_n^{(t+1)} - U_m^{(t)}). \end{aligned}$$

Then, we obtain the relation

$$\begin{aligned} \Delta_m - U_m^{(t)} - U_m^{(t+1)} &= \Lambda_{n+1}^{(t)} - U_m^{(t)} - \min(\Lambda_{n+1}^{(t)} - U_m^{(t)}, D_n^{(t+1)} - U_m^{(t)}) \\ &= -\min(0, D_n^{(t+1)} - \Lambda_{n+1}^{(t)}) \\ &= \max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)}), \end{aligned}$$

where we have used the formula (4.10). This relation implies that the term $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ denotes the size of interspace inserted between the n-th soliton at time t and the n-th one at time $t+1$.

Once we notice the role of the terms $\max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$, we can now clarify the meaning of equations (4.12c) and (4.12d). Since the term $E_{n+1}^{(t)} - \max(0, \Lambda_{n+1}^{(t)} - D_n^{(t+1)})$ in (4.12c) denotes the difference between the size of the inserted space and the one of the $(n+1)$-th empty block, $Q_n^{(t+1)}$ should be determined by (4.12c). Similarly, $E_n^{(t+1)}$ should be determined by (4.12d). It is also true for $n = 0$ and 1, then the proof is completed by induction. ■

4.3 Finite Toda representation of the box–ball system with variable box capacity and carrier capacity

Next, we construct the finite Toda representation of the BBS with both box capacity and carrier capacity from two time evolution maps: the size limit map and the recovery map. This is the same as for the construction of the Euler representation explained in Section 4.1. Figure 4.6 shows an example.

$\tilde{Q}_0^{(t)}\ \tilde{E}_1^{(t)}\ \tilde{Q}_1^{(t)}\ \tilde{E}_2^{(t)}\ \tilde{Q}_2^{(t)}$

```
t=0: |..1|11111|11.|....1|111|.....|...|...11|...|.....|...|.....|...|.....|...|  8 5 4 11 2
  1: |...|.....|..1|1111.|...|11111|...|.....|.11|.....|...|.....|...|.....|...|  5 4 5  9 2
  2: |...|.....|...|....1|111|.....|111|111..|...|...11|...|.....|...|.....|...|  4 5 6  8 2
  3: |...|.....|...|.....|...|.1111|...|...11|111|1....|.11|.....|...|.....|...|  4 6 6  5 2
  4: |...|.....|...|.....|...|.....|111|1....|...|.1111|1..|..111|...|.....|...|  4 8 5  4 3
  5: |...|.....|...|.....|...|.....|...|..111|1..|.....|.11|11...|111|1....|...|  4 8 4  3 4
  6: |...|.....|...|.....|...|.....|...|.....|.11|11...|...|...11|...|.1111|11.|  4 9 2  4 6
```

$Q_0^{(t)}\ E_1^{(t)}\ Q_1^{(t)}\ E_2^{(t)}\ Q_2^{(t)}$

```
t=0: |..1|11111|11.|....1|111|.....|...|...11|...|.....|...|.....|...|.....|...|  8 5 4 11 2
  1: |...|.....|111|1111.|...|11111|...|.....|.11|.....|...|.....|...|.....|...|  7 4 5  9 2
  2: |...|.....|...|...11|111|1....|111|111..|...|...11|...|.....|...|.....|...|  6 4 6  8 2
  3: |...|.....|...|.....|...|.1111|...|.1111|111|1....|.11|.....|...|.....|...|  4 4 8  5 2
  4: |...|.....|...|.....|...|.....|111|1....|..1|11111|1..|..111|...|.....|...|  4 6 7  4 3
  5: |...|.....|...|.....|...|.....|...|..111|1..|.....|111|111..|111|1....|...|  4 7 6  2 4
  6: |...|.....|...|.....|...|.....|...|.....|.11|11...|...|...11|1..|11111|11.|  4 9 3  2 7
```

Figure 4.6: Example of the expansion map for the BBS with box capacity Δ_n and carrier capacity $S^{(t)} = 6$ for $t > 0$. This is obtained from the example in Figure 4.2 via the expansion map.

In the next theorem, we use the following notations:

- $Q_n^{(t)}$, $E_n^{(t)}$: the size of the n-th soliton and the one of the n-th empty block at time t, respectively;
- $\tilde{Q}_n^{(t+1)}$, $\tilde{E}_n^{(t+1)}$: the size of the n-th soliton and the one of the n-th empty block after the size limit process from time t to $t+1$;
- $\tilde{C}_n^{(t+1)}$, $\tilde{D}_n^{(t+1)}$: some quantities which will be explained in the proof of the next theorem in detail;
- $K_n^{(t)}$, $\Lambda_n^{(t)}$: the capacity of the box which contains the leftmost segment of the n-th soliton and the one of the n-th empty block at time t, respectively.

Theorem 4.3. *Let the variables $Q_n^{(t)}$, $E_n^{(t)}$, $\tilde{Q}_n^{(t+1)}$, $\tilde{E}_n^{(t+1)}$, $\tilde{C}_n^{(t+1)}$ and $\tilde{D}_n^{(t+1)}$ denote the quantities of the BBS as explained in the above. Then, the time evolution of the BBS with box capacity $K_n^{(t)}$*

and $\Lambda_n^{(t)}$*, and carrier capacity* $S^{(t+1)}$ *is given by*

$$\tilde{C}_0^{(t+1)} = K_0^{(t)}, \tag{4.13a}$$
$$\tilde{C}_n^{(t+1)} = \min(\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + K_n^{(t)}, S^{(t+1)}), \tag{4.13b}$$
$$\tilde{D}_n^{(t+1)} = \min(\tilde{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}, S^{(t+1)}), \tag{4.13c}$$
$$\tilde{Q}_n^{(t+1)} = \min\left(E_{n+1}^{(t)} - \max(0, \Lambda_{n+1}^{(t)} - \tilde{D}_n^{(t+1)}), \tilde{D}_n^{(t+1)}\right), \tag{4.13d}$$
$$\begin{aligned}\tilde{E}_n^{(t+1)} = {} & E_n^{(t)} - \tilde{Q}_{n-1}^{(t+1)} + Q_n^{(t)} \\ & - \max(0, \Lambda_n^{(t)} - \tilde{D}_{n-1}^{(t+1)}) + \max(0, \Lambda_{n+1}^{(t)} - \tilde{D}_n^{(t+1)}),\end{aligned} \tag{4.13e}$$
$$Q_n^{(t+1)} = Q_n^{(t)} + \tilde{C}_n^{(t+1)} - \tilde{C}_{n+1}^{(t+1)} - K_n^{(t)} + K_{n+1}^{(t)}, \tag{4.13f}$$
$$E_n^{(t+1)} = \tilde{E}_n^{(t+1)} + \tilde{Q}_{n-1}^{(t+1)} - Q_n^{(t)} - \tilde{D}_{n-1}^{(t+1)} + \tilde{D}_n^{(t+1)}, \tag{4.13g}$$
$$E_0^{(t)} = E_N^{(t)} = \tilde{E}_0^{(t)} = \tilde{E}_N^{(t)} = +\infty \quad \textit{for all } t, \tag{4.13h}$$

where the carrier capacity $S^{(t+1)}$ *must be chosen to satisfy the condition* $K_n^{(t)} \leq S^{(t+1)}$ *for all n and t.*

When the quantities $\{Q_n^{(t)}\}_{n=0}^{N-1}$ and $\{E_n^{(t)}\}_{n=1}^{N-1}$ are given, first we can calculate $\tilde{D}_0^{(t+1)}$ using (4.13a) and (4.13c). Next, we can calculate $\tilde{Q}_0^{(t+1)}$ by (4.13d), $\tilde{C}_1^{(t+1)}$ by (4.13b), $\tilde{D}_1^{(t+1)}$ by (4.13c). In a repetitive manner, we can obtain the quantities $\{\tilde{Q}_n^{(t+1)}\}_{n=0}^{N-1}$, $\{\tilde{C}_n^{(t+1)}\}_{n=0}^{N}$ and $\{\tilde{D}_n^{(t+1)}\}_{n=0}^{N-1}$. Finally, we can calculate the quantities $\{\tilde{E}_n^{(t+1)}\}_{n=1}^{N-1}$, $\{Q_n^{(t+1)}\}_{n=0}^{N-1}$ and $\{E_n^{(t+1)}\}_{n=1}^{N-1}$ by (4.13e), (4.13f) and (4.13g), respectively. Hence, the time evolution is determined by (4.13).

If $S^{(t+1)} = +\infty$, then (4.13b) and (4.13c) reduce to $\tilde{C}_n^{(t+1)} = \tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + K_n^{(t)}$ and $\tilde{D}_n^{(t+1)} = \tilde{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}$, respectively. Thus, we have the equation $\tilde{D}_n^{(t+1)} = \tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + Q_n^{(t)}$ and, substituting them into (4.13f) and (4.13g), we obtain $Q_n^{(t+1)} = \tilde{Q}_n^{(t+1)}$ and $E_n^{(t+1)} = \tilde{E}_n^{(t+1)}$. Hence, in this case, the ultradiscrete system (4.13) reduces to the system (4.12). We can therefore say that the system (4.13) is a generalization of the system (4.12).

Proof. Let us show that equations (4.13a)–(4.13e) and (4.13f)–(4.13g) describe the size limit process and the recovery process, respectively.

First, we consider the size limit process. Equations (4.13d) and (4.13e) have the same forms as of (4.12c) and (4.12d). Thus, we shall investigate the variables $\tilde{C}_n^{(t+1)}$ and $\tilde{D}_n^{(t+1)}$ which are defined by (4.13b) and (4.13c). Suppose that the carrier capacity is chosen as $K_n^{(t)} \leq S^{(t+1)} < +\infty$, $\tilde{Q}_j^{(t+1)}$, $j = 0, 1, \ldots, n-1$, and $\tilde{E}_j^{(t+1)}$, $j = 1, 2, \ldots, n-1$, are given, and the quantity $\tilde{D}_{n-1}^{(t+1)}$ denotes the number of balls which the carrier holds after getting $Q_{n-1}^{(t)}$ balls from boxes and restricting the number of the balls in the carrier to $S^{(t+1)}$ balls. Since the inequality $\tilde{Q}_{n-1}^{(t+1)} \leq \tilde{D}_{n-1}^{(t+1)}$ holds from (4.13d), it is sufficient to consider the following two cases: the case of which the carrier drops off all balls temporarily ($\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} = 0$) and the case of which the carrier has balls just before getting $Q_n^{(t)}$ balls ($\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} > 0$).

(i) If $\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} = 0$, then $\tilde{C}_n^{(t+1)} = \min(\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + K_n^{(t)}, S^{(t+1)}) = K_n^{(t)}$ holds from the assumption. We should note that, in this case, the number of balls which the carrier holds is zero temporarily before getting $Q_n^{(t)}$ balls. Thus, from (4.13c), we have $\tilde{D}_n^{(t+1)} = \min(Q_n^{(t)}, S^{(t+1)})$, which indicates that the quantity $\tilde{D}_n^{(t+1)}$ is again the number of balls which the carrier holds after getting $Q_n^{(t)}$ balls and restricting the number of the balls in the carrier to $S^{(t+1)}$ balls.

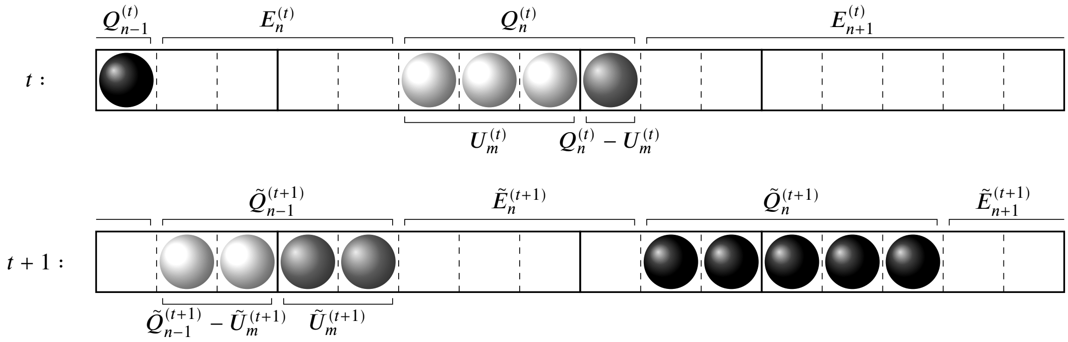


Figure 4.7: Illustration of the size limit process when the carrier parameter $S^{(t+1)} = 6$ and $\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} > 0$. We can see that $\tilde{C}_n^{(t+1)}$ denotes the number of balls which the carrier holds after getting $U_m^{(t)}$ balls (white balls) and restricting the number of the balls in the carrier to $S^{(t+1)}$, and $\tilde{D}_n^{(t+1)}$ denotes the number of balls which the carrier holds after getting $Q_n^{(t)} - U_m^{(t)}$ balls (gray balls) and restricting the number of the balls in the carrier to $S^{(t+1)}$ balls.

Table 4.1: Change of the number of balls which the carrier holds.

State of the carrier	The number of balls which the carrier holds
$\vdots$	$\vdots$
Getting $Q_{n-1}^{(t)}$ balls	$\tilde{D}_{n-1}^{(t+1)}$
Putting $\tilde{Q}_{n-1}^{(t+1)} - \tilde{U}_m^{(t+1)}$ balls	$\tilde{D}_{n-1}^{(t+1)} - (\tilde{Q}_{n-1}^{(t+1)} - \tilde{U}_m^{(t+1)})$
Getting $U_m^{(t)}$ balls	$\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + K_n^{(t)}$
Size limit to $S^{(t+1)}$ balls	$\tilde{C}_n^{(t+1)}$
Putting $\tilde{U}_m^{(t+1)}$ balls	$\tilde{C}_n^{(t+1)} - \tilde{U}_m^{(t+1)}$
Getting $Q_n^{(t)} - U_m^{(t)}$ balls	$\tilde{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}$
Size limit to $S^{(t+1)}$ balls	$\tilde{D}_n^{(t+1)}$
$\vdots$	$\vdots$

(ii) The case of $\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} > 0$. Let m be the index of the box which contains the leftmost segment of the n-th soliton at time t. Under the assumption, in the terms of the variables of the Euler representation (4.6), $\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} > 0$ implies that $\tilde{U}_m^{(t+1)} = \Delta_m - U_m^{(t)}$ should hold (see Figure 4.7). Now $\Delta_m = K_n^{(t)}$ by definition. Hence, we can write (4.13b) and (4.13c) as $\tilde{C}_n^{(t+1)} = \min(\tilde{D}_{n-1}^{(t+1)} - (\tilde{Q}_{n-1}^{(t+1)} - \tilde{U}_m^{(t+1)}) + U_m^{(t)}, S^{(t+1)})$ and $\tilde{D}_n^{(t+1)} = \min(\tilde{C}_n^{(t+1)} + (Q_n^{(t)} - U_m^{(t)}) - \tilde{U}_m^{(t+1)}, S^{(t+1)})$, respectively. Therefore, the quantity $\tilde{D}_n^{(t+1)}$ is again the number of balls which the carrier holds after getting $Q_n^{(t)}$ balls and restricting the number of the balls in the carrier to $S^{(t+1)}$ balls. We can summarize the change of the number of balls which the carrier holds as Table 4.1.

Thus, together with the proof of Theorem 4.3, it is proved that (4.13d)–(4.13c) describe the size limit process by induction.

Furthermore, since the number of balls removed by the size limit process are given by $(\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + K_n^{(t)}) - \tilde{C}_n^{(t+1)}$ and $(\tilde{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)}) - \tilde{D}_n^{(t+1)}$, we obtain the equations of the recovery process

$$\begin{aligned}
Q_n^{(t+1)} &= \tilde{Q}_n^{(t+1)} + (\tilde{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)} - \tilde{D}_n^{(t+1)}) + (\tilde{D}_n^{(t+1)} - \tilde{Q}_n^{(t+1)} + K_{n+1}^{(t)} - \tilde{C}_{n+1}^{(t+1)}) \\
&= Q_n^{(t)} + \tilde{C}_n^{(t+1)} - \tilde{C}_{n+1}^{(t+1)} - K_n^{(t)} + K_{n+1}^{(t)}, \\
E_n^{(t+1)} &= \tilde{E}_n^{(t+1)} - (\tilde{D}_{n-1}^{(t+1)} - \tilde{Q}_{n-1}^{(t+1)} + K_n^{(t)} - \tilde{C}_n^{(t+1)}) - (\tilde{C}_n^{(t+1)} + Q_n^{(t)} - K_n^{(t)} - \tilde{D}_n^{(t+1)}) \\
&= \tilde{E}_n^{(t+1)} + \tilde{Q}_{n-1}^{(t+1)} - Q_n^{(t)} - \tilde{D}_{n-1}^{(t+1)} + \tilde{D}_n^{(t+1)},
\end{aligned}$$

which lead to equations (4.13f) and (4.13g), and the proof is completed. ■

Remark 4.4. Furthermore, the variables $X_0^{(t)}$ and $\tilde{X}_0^{(t)}$, which denote the index of the leftmost segment of the 0th soliton at time t satisfy the equations

$$\tilde{X}_0^{(t+1)} = X_0^{(t)} + Q_0^{(t)} + \max(0, \Lambda_1^{(t)} - \tilde{D}_0^{(t+1)}), \tag{4.14a}$$

$$X_0^{(t+1)} = \tilde{X}_0^{(t+1)} - Q_0^{(t)} + \tilde{D}_0^{(t+1)} = X_0^{(t)} + \max(\tilde{D}_0^{(t+1)}, \Lambda_1^{(t)}). \tag{4.14b}$$

4.4 Particular solutions to the case of fixed box capacity

In this section, we discuss a particular solution to the ultradiscrete system (4.13) with a special condition: all boxes have constant capacity Δ.

Let us consider the bilinear equations

$$\tilde{\tau}_n^{0,t+1}\tau_n^{1,t-1} = \delta\tau_{n+1}^{0,t-1}\tilde{\tau}_{n-1}^{1,t+1} + \tau_n^{0,t}\tilde{\tau}_n^{1,t}, \tag{4.15a}$$

$$\delta\tilde{\tau}_n^{0,t}\tau_n^{1,t} = (\delta - s^{(t)})\tau_n^{0,t}\tilde{\tau}_n^{1,t} + s^{(t)}\tilde{\tau}_n^{0,t+1}\tau_n^{1,t-1}, \tag{4.15b}$$

$$\tau_{n+1}^{0,t}\tilde{\tau}_n^{1,t} = \tilde{\tau}_{n+1}^{0,t}\tau_n^{1,t} + s^{(t)}\tilde{\tau}_n^{0,t+1}\tau_{n+1}^{1,t-1}, \tag{4.15c}$$

$$(\delta - s^{(t)})\tilde{\tau}_n^{0,t+1}\tau_{n+1}^{1,t-1} + \tau_{n+1}^{0,t}\tilde{\tau}_n^{1,t} = \tau_{n+1}^{0,t-1}\tilde{\tau}_n^{1,t+1}, \tag{4.15d}$$

where δ is a constant and $s^{(t)}$ is a parameter depending on t. We introduce the dependent variables

$$\begin{aligned}
q_n^{(t)} &= \frac{\tilde{\tau}_{n+1}^{0,t+1}\tau_n^{1,t}}{\tilde{\tau}_n^{0,t+1}\tau_{n+1}^{1,t}}, & \tilde{q}_n^{(t)} &= \frac{\delta}{\delta - s^{(t)}}\frac{\tilde{\tau}_{n+1}^{0,t+1}\tilde{\tau}_n^{1,t}}{\tilde{\tau}_n^{0,t+1}\tilde{\tau}_{n+1}^{1,t}}, \\
e_n^{(t)} &= \delta^2\frac{\tau_{n+1}^{0,t}\tilde{\tau}_{n-1}^{1,t+1}}{\tau_n^{0,t}\tilde{\tau}_n^{1,t+1}}, & \tilde{e}_n^{(t)} &= \delta(\delta - s^{(t)})\frac{\tilde{\tau}_{n+1}^{0,t}\tilde{\tau}_{n-1}^{1,t+1}}{\tilde{\tau}_n^{0,t}\tilde{\tau}_n^{1,t+1}}, \\
\tilde{c}_n^{(t)} &= \delta\frac{\tilde{\tau}_n^{0,t}\tau_n^{1,t}}{\tilde{\tau}_n^{0,t+1}\tau_n^{1,t-1}}, & \tilde{d}_n^{(t)} &= \frac{\delta}{\delta - s^{(t)}}\frac{\tau_{n+1}^{0,t}\tilde{\tau}_n^{1,t}}{\tilde{\tau}_n^{0,t+1}\tau_{n+1}^{1,t-1}}.
\end{aligned}$$

Then, (4.15d) yields the relation

$$1 + \delta^{-1}d_n^{(t)} = (\delta - s^{(t)})^{-1}\frac{\tau_{n+1}^{0,t-1}\tilde{\tau}_n^{1,t+1}}{\tilde{\tau}_n^{0,t+1}\tau_{n+1}^{1,t-1}}.$$

Furthermore, (4.15a)–(4.15c) yield the equations

$$\tilde{q}_n^{(t+1)} = e_{n+1}^{(t)}(1 + \delta^{-1}\tilde{d}_n^{(t+1)}) + \tilde{d}_n^{(t+1)}, \tag{4.16a}$$

$$\tilde{c}_n^{(t+1)} = (\delta - s^{(t+1)})\frac{\tilde{d}_{n-1}^{(t+1)}}{\tilde{q}_{n-1}^{(t+1)}} + s^{(t+1)}, \tag{4.16b}$$

$$\tilde{d}_n^{(t+1)} = (\delta - s^{(t+1)})^{-1}\tilde{c}_n^{(t+1)} q_n^{(t)} + \frac{s^{(t+1)}}{1-\delta^{-1}s^{(t+1)}}, \tag{4.16c}$$

and the identities

$$\tilde{e}_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{\tilde{q}_{n-1}^{(t+1)}} \frac{1+\delta^{-1}\tilde{d}_{n-1}^{(t+1)}}{1+\delta^{-1}\tilde{d}_n^{(t+1)}}, \tag{4.16d}$$

$$q_n^{(t+1)} = q_n^{(t)} \frac{\tilde{c}_n^{(t+1)}}{\tilde{c}_{n+1}^{(t+1)}}, \tag{4.16e}$$

$$e_n^{(t+1)} = \tilde{e}_n^{(t+1)} \frac{\tilde{q}_{n-1}^{(t+1)}\tilde{d}_n^{(t+1)}}{q_n^{(t)}\tilde{d}_{n-1}^{(t+1)}}, \tag{4.16f}$$

hold. In addition, we impose the finite lattice condition

$$e_0^{(t)} = e_N^{(t)} = \tilde{e}_0^{(t)} = \tilde{e}_N^{(t)} = 0. \tag{4.16g}$$

In the bilinear equations (4.15), this condition implies

$$\tau_{-1}^{k,t} = \tau_{N+1}^{k,t} = \tilde{\tau}_{-1}^{k,t} = \tilde{\tau}_{N+1}^{k,t} = 0.$$

We assume that the constant δ and the parameter $s^{(t)}$ satisfy the condition $0 < s^{(t)} < \delta$ for all $t \in \mathbb{Z}$. Then, putting $q_n^{(t)} = \mathrm{e}^{-Q_n^{(t)}/\epsilon}$, $e_n^{(t)} = \mathrm{e}^{-E_n^{(t)}/\epsilon}$, $\tilde{q}_n^{(t)} = \mathrm{e}^{-\tilde{Q}_n^{(t)}/\epsilon}$, $\tilde{e}_n^{(t)} = \mathrm{e}^{-\tilde{E}_n^{(t)}/\epsilon}$, $\tilde{c}_n^{(t)} = \mathrm{e}^{-\tilde{C}_n^{(t)}/\epsilon}$, $\tilde{d}_n^{(t)} = \mathrm{e}^{-\tilde{D}_n^{(t)}/\epsilon}$, $\delta = \mathrm{e}^{-\Delta/\epsilon}$ into (4.16) and taking a limit $\epsilon \to +0$, we obtain the ultradiscrete system (4.13) with the condition $K_n^{(t)} = \Lambda_n^{(t)} = \Delta \le S^{(t+1)}$ for all $n, t \in \mathbb{Z}$.

The following theorem is proved by using a determinant identity called the Plücker relation.

Theorem 4.5. *A particular solution to the bilinear equations* (4.15) *with the semi-infinite lattice condition* $\tau_{-1}^{k,t} = \tilde{\tau}_{-1}^{k,t} = 0$ *for all* $k, t \in \mathbb{Z}$ *is given by the Hankel determinants*

$$\tau_n^{k,t} = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ |\xi_{k+i+j}^{(t)}|_{0 \le i,j \le n-1} & \text{if } n > 0, \end{cases} \tag{4.17a}$$

$$\tilde{\tau}_n^{k,t} = \begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ |\tilde{\xi}_{k+i+j}^{(t)}|_{0 \le i,j \le n-1} & \text{if } n > 0, \end{cases} \tag{4.17b}$$

where $\xi_n^{(t)}$ *and* $\tilde{\xi}_n^{(t)}$ *are arbitrary functions satisfying the dispersion relation*

$$\tilde{\xi}_n^{(t+1)} = -\delta\xi_{n+1}^{(t)} + \xi_n^{(t)} = (s^{(t)} - \delta)\tilde{\xi}_{n+1}^{(t)} + \tilde{\xi}_n^{(t)}, \quad n = 0, 1, \dots. \tag{4.18}$$

Hereafter, we choose the arbitrary functions as

$$\xi_n^{(t)} = \sum_{i=0}^{N-1} \frac{\eta_i^{(t)}}{x_i (x_i + \delta)^n}, \quad \tilde{\xi}_n^{(t)} = \sum_{i=0}^{N-1} \frac{\eta_i^{(t-1)}}{(x_i + \delta)^{n+1}}, \quad \eta_i^{(t)} := \frac{w_i \prod_{j=0}^{t} (x_i + s^{(j)})}{(x_i + \delta)^t}, \tag{4.19}$$

where x_i and w_i, $i = 0, 1, \dots, N-1$, are some constants. Then, the dispersion relation (4.18) is satisfied and the finite lattice condition $\tau_{-1}^{k,t} = \tau_{N+1}^{k,t} = \tilde{\tau}_{-1}^{k,t} = \tilde{\tau}_{N+1}^{k,t} = 0$ holds for all $k, t \in \mathbb{Z}$.

Substituting (4.19) to (4.17) and expanding the Hankel determinants using the Binet–Cauchy formula, we obtain

$$\tau_n^{k,t} = \sum_{0\le r_0<r_1<\cdots<r_{n-1}\le N-1}\left(\left(\prod_{0\le i<j\le n-1}\frac{x_{r_i}-x_{r_j}}{(x_{r_i}+\delta)(x_{r_j}+\delta)}\right)^2\prod_{i=0}^{n-1}\frac{w_{r_i}\prod_{j=0}^{t}(x_{r_i}+s^{(j)})}{x_{r_i}(x_{r_i}+\delta)^{t+k}}\right),$$

$$\tilde{\tau}_n^{k,t} = \sum_{0\le r_0<r_1<\cdots<r_{n-1}\le N-1}\left(\left(\prod_{0\le i<j\le n-1}\frac{x_{r_i}-x_{r_j}}{(x_{r_i}+\delta)(x_{r_j}+\delta)}\right)^2\prod_{i=0}^{n-1}\frac{w_{r_i}\prod_{j=0}^{t-1}(x_{r_i}+s^{(j)})}{(x_{r_i}+\delta)^{t+k}}\right),$$

for $n = 1,2,\dots,N$. These expressions can be ultradiscretized directly: putting $x_n = \mathrm{e}^{-X_n/\epsilon}$, $w_n = \mathrm{e}^{-W_n/\epsilon}$, $\tau_n^{k,t} = \mathrm{e}^{-T_n^{k,t}/\epsilon}$, $\tilde{\tau}_n^{k,t} = \mathrm{e}^{-\tilde{T}_n^{k,t}/\epsilon}$, and taking a limit $\epsilon \to +0$, we obtain the next theorem.

Theorem 4.6. *A particular solution to the ultradiscrete system* (4.13) *with the condition $K_n^{(t)} = \Lambda_n^{(t)} = \Delta \le S^{(t+1)}$ for all $n, t \in \mathbb{Z}$ is given by*

$$\begin{aligned}
Q_n^{(t)} &= \tilde{T}_{n+1}^{0,t+1} - \tilde{T}_n^{0,t+1} + T_n^{1,t} - T_{n+1}^{1,t}, & \tilde{Q}_n^{(t)} &= \tilde{T}_{n+1}^{0,t+1} - \tilde{T}_n^{0,t+1} + \tilde{T}_n^{1,t} - \tilde{T}_{n+1}^{1,t},\\
E_n^{(t)} &= T_{n+1}^{0,t} - T_n^{0,t} + \tilde{T}_{n-1}^{1,t+1} - \tilde{T}_n^{1,t+1} + 2\Delta, & \tilde{E}_n^{(t)} &= \tilde{T}_{n+1}^{0,t} - \tilde{T}_n^{0,t} + \tilde{T}_{n-1}^{1,t+1} - \tilde{T}_n^{1,t+1} + 2\Delta,\\
\tilde{C}_n^{(t)} &= \tilde{T}_n^{0,t} - \tilde{T}_n^{0,t+1} + T_n^{1,t} - T_n^{1,t-1} + \Delta, & \tilde{D}_n^{(t)} &= T_{n+1}^{0,t} - \tilde{T}_n^{0,t+1} + \tilde{T}_n^{1,t} - T_{n+1}^{1,t-1},
\end{aligned}$$

$$\begin{aligned}
T_n^{k,t} &= \min_{0\le r_0<r_1<\cdots<r_{n-1}\le N-1}\left(\sum_{i=0}^{n-1}\Big(W_{r_i} + (2(n-1-i)-1)X_{r_i}\right.\\
&\qquad \left.-(2(n-1)+t+k)\min(X_{r_i},\Delta) + \sum_{j=0}^{t}\min(X_{r_i},S^{(j)})\Big)\right), \quad n = 1,2,\dots,N,\\
\tilde{T}_n^{k,t} &= \min_{0\le r_0<r_1<\cdots<r_{n-1}\le N-1}\left(\sum_{i=0}^{n-1}\Big(W_{r_i} + 2(n-1-i)X_{r_i}\right.\\
&\qquad \left.-(2(n-1)+t+k)\min(X_{r_i},\Delta) + \sum_{j=0}^{t-1}\min(X_{r_i},S^{(j)})\Big)\right), \quad n = 1,2,\dots,N,\\
T_{-1}^{k,t} &= T_{N+1}^{k,t} = \tilde{T}_{-1}^{k,t} = \tilde{T}_{N+1}^{k,t} = +\infty, \quad T_0^{k,t} = \tilde{T}_0^{k,t} = 0,
\end{aligned}$$

where X_i and W_i, $i = 0,1,\dots,N-1$, are some constants satisfying $X_0 \le X_1 \le \dots \le X_{N-1}$.

Remark 4.7. There exists a Bäcklund transformation from the discrete system (4.16) to the nd-Toda lattice:

$$\begin{aligned}
\mathsf{q}_n^{(t)} &= \frac{\delta^{-1}q_n^{(t)}}{\delta(1+\delta^{-1}q_n^{(t)})(1+\delta^{-1}e_n^{(t)})}, & \mathsf{e}_n^{(t)} &= \frac{\delta^{-1}e_n^{(t)}}{\delta(1+\delta^{-1}q_{n-1}^{(t)})(1+\delta^{-1}e_n^{(t)})},\\
\tilde{\mathsf{q}}_n^{(t)} &= \frac{\delta^{-1}\tilde{q}_n^{(t)}}{(\delta-s^{(t)})(1+\delta^{-1}\tilde{q}_n^{(t)})(1+\delta^{-1}\tilde{e}_n^{(t)})}, & \tilde{\mathsf{e}}_n^{(t)} &= \frac{\delta^{-1}\tilde{e}_n^{(t)}}{(\delta-s^{(t)})(1+\delta^{-1}\tilde{q}_{n-1}^{(t)})(1+\delta^{-1}\tilde{e}_n^{(t)})}.
\end{aligned}$$

In fact, these variables have τ-function expressions

$$\mathsf{q}_n^{(t)} = \delta^{-1}\frac{\tau_n^{0,t}\tilde{\tau}_{n+1}^{0,t+1}}{\tau_{n+1}^{0,t}\tilde{\tau}_n^{0,t+1}}, \quad \tilde{\mathsf{q}}_n^{(t)} = (\delta - s^{(t)})^{-1}\frac{\tilde{\tau}_n^{0,t}\tilde{\tau}_{n+1}^{0,t+1}}{\tilde{\tau}_{n+1}^{0,t}\tilde{\tau}_n^{0,t+1}},$$

$$e_n^{(t)} = \delta \frac{\tau_{n+1}^{0,t} \tilde{\tau}_{n-1}^{0,t+1}}{\tau_n^{0,t} \tilde{\tau}_n^{0,t+1}}, \qquad \tilde{e}_n^{(t)} = (\delta - s^{(t)}) \frac{\tilde{\tau}_{n+1}^{0,t} \tilde{\tau}_{n-1}^{0,t+1}}{\tilde{\tau}_n^{0,t} \tilde{\tau}_n^{0,t+1}}.$$

Since the bilinear equations

$$\begin{aligned}
&\tau_n^{0,t-1} \tilde{\tau}_n^{0,t+1} = \delta(\delta - s^{(t)}) \tau_{n+1}^{0,t-1} \tilde{\tau}_{n-1}^{0,t+1} + \tau_n^{0,t} \tilde{\tau}_n^{0,t}, \\
&\delta \tilde{\tau}_n^{0,t} \tau_{n+1}^{0,t} = (\delta - s^{(t)}) \tau_n^{0,t} \tilde{\tau}_{n+1}^{0,t} + s^{(t)} \tau_{n+1}^{0,t-1} \tilde{\tau}_n^{0,t+1}
\end{aligned}$$

hold (these are proved by using the Plücker relation), we have the equations

$$\tilde{q}_n^{(t+1)} = e_{n+1}^{(t)} + \tilde{d}_n^{(t+1)}, \quad \tilde{d}_n^{(t+1)} = \tilde{d}_{n-1}^{(t+1)} \frac{q_n^{(t)}}{\tilde{q}_{n-1}^{(t+1)}} + \sigma_{t+1},$$

where

$$\tilde{d}_n^{(t)} := \delta - (s^{(t)})^{-1} \frac{\tilde{\tau}_n^{0,t} \tau_{n+1}^{0,t}}{\tau_{n+1}^{0,t-1} \tilde{\tau}_n^{0,t+1}}, \quad \sigma_t := \frac{\delta^{-1} s^{(t)}}{\delta - s^{(t)}}.$$

Additionally, we have the identities

$$\tilde{e}_n^{(t+1)} = e_n^{(t)} \frac{q_n^{(t)}}{\tilde{q}_{n-1}^{(t+1)}}, \quad q_n^{(t+1)} = \tilde{q}_n^{(t+1)} \frac{\tilde{d}_{n-1}^{(t+1)} q_n^{(t)}}{\tilde{d}_n^{(t+1)} \tilde{q}_{n-1}^{(t+1)}}, \quad e_n^{(t+1)} = \tilde{e}_n^{(t+1)} \frac{\tilde{d}_n^{(t+1)} \tilde{q}_{n-1}^{(t+1)}}{\tilde{d}_{n-1}^{(t+1)} q_n^{(t)}}.$$

Eliminating $\tilde{d}_n^{(t+1)}$ from these equations, we obtain the modified nd-Toda lattice

$$\begin{aligned}
&\tilde{q}_n^{(t+1)} + \tilde{e}_n^{(t+1)} = q_n^{(t)} + e_{n+1}^{(t)} + \sigma_{t+1}, \qquad \tilde{q}_{n-1}^{(t+1)} \tilde{e}_n^{(t+1)} = q_n^{(t)} e_n^{(t)}, \\
&q_n^{(t+1)} + e_{n+1}^{(t+1)} = \tilde{q}_n^{(t+1)} + \tilde{e}_{n+1}^{(t+1)} - \sigma_{t+1}, \quad q_n^{(t+1)} e_n^{(t+1)} = \tilde{q}_n^{(t+1)} \tilde{e}_n^{(t+1)},
\end{aligned}$$

and the finite lattice condition is given by

$$e_0^{(t)} = e_N^{(t)} = \tilde{e}_0^{(t)} = \tilde{e}_N^{(t)} = 0 \quad \text{for all } t.$$

Chapter 5

Finite $\mathrm{R_{II}}$ chain and Generalized Eigenvalue Algorithm

In Chapter 2, we reviewed that the eigenvalue problem of a tridiagonal matrix determines a monic orthogonal polynomial sequence, and that the recurrence relations of the dqds algorithm are derived as the time evolution equations of the nonautonomous discrete integrable finite lattices associated with the monic orthogonal polynomial sequence. In this chapter, we extend the theory above for tridiagonal matrix pencils and their associated nonautonomous discrete integrable lattices.

5.1 $\mathrm{R_{II}}$ polynomials

Let us consider two tridiagonal semi-infinite matrices in the following forms:

$$A^{(t)} = \begin{pmatrix} \alpha_0^{(t)} & \kappa_t & & & \\ \lambda_1\beta_1^{(t)} & \alpha_1^{(t)} & \kappa_{t+1} & & \\ & \lambda_2\beta_2^{(t)} & \alpha_2^{(t)} & \kappa_{t+2} & \\ & & \lambda_3\beta_3^{(t)} & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \quad \alpha_n^{(t)}, \kappa_{t+n}, \lambda_n \in \mathbb{R}, \quad \beta_n^{(t)} \in \mathbb{R} - \{0\},$$

$$B^{(t)} = \begin{pmatrix} \tilde{\alpha}_0^{(t)} & 1 & & & \\ \beta_1^{(t)} & \tilde{\alpha}_1^{(t)} & 1 & & \\ & \beta_2^{(t)} & \tilde{\alpha}_2^{(t)} & 1 & \\ & & \beta_3^{(t)} & \ddots & \ddots \\ & & & \ddots & \ddots \end{pmatrix}, \qquad \tilde{\alpha}_n^{(t)} \in \mathbb{R}.$$

Let $A_n^{(t)}$ and $B_n^{(t)}$ denote the n-th order leading principal submatrices of $A^{(t)}$ and $B^{(t)}$, respectively. We now define a polynomial sequence $\{\varphi_n^{(t)}(x)\}_{n=0}^{\infty}$ by

$$\varphi_0^{(t)}(x) := 1, \quad \varphi_n^{(t)}(x) := \det(xB_n^{(t)} - A_n^{(t)}), \quad n = 1, 2, 3, \dots .$$

The polynomial $\varphi_n^{(t)}(x)$ is a monic polynomial of degree n. In the same manner as in the case of monic orthogonal polynomials in Section 2.3, we obtain the three-term recurrence relation

$$\varphi_{n+1}^{(t)}(x) = (\tilde{\alpha}_n^{(t)} x - \alpha_n^{(t)})\varphi_n^{(t)}(x) - \beta_n^{(t)}(x - \kappa_{t+n-1})(x - \lambda_n)\varphi_{n-1}^{(t)}(x), \quad n = 0, 1, 2, \dots , \tag{5.1}$$

where we set $\beta_0^{(t)} := 0$ and $\varphi_{-1}^{(t)}(x) := 0$. We will assume in what follows that all the parameters κ_{t+k} and λ_k, $k = 0, 1, 2, \dots$, are not zeros of the polynomial $\varphi_n^{(t)}(x)$ for all $n \in \mathbb{N}$. The polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^{\infty}$ are called the *R_{II} polynomials* with respect to $\mathscr{L}^{(t)}$, introduced by Ismail and Masson [34].

We introduce the notations

$$K_k^{(t)}(x) := \prod_{j=0}^{k-1}(x - \kappa_{t+j}), \quad K_k(x) := K_k^{(0)}(x) = \prod_{j=0}^{k-1}(x - \kappa_j), \quad L_l(x) := \prod_{j=1}^{l}(x - \lambda_j),$$

and $\mathscr{D}(\mathscr{L}^{(t)})$ a linear space spanned by the rational functions $\frac{x^m}{K_k^{(t)}(x)L_l(x)}$, $k, l = 0, 1, 2, \dots$; $m = 0, 1, \dots, k + l$. The following Favard type theorem is proved.

Theorem 5.1 (Favard type theorem for the $\mathrm{R_{II}}$ polynomials [34]). *Let $\{\varphi_n^{(t)}(x)\}_{n=0}^{\infty}$ be the R_{II} polynomials. For any nonzero constants $h_0^{(t)}$ and $h_1^{(t)}$, which satisfy $h_0^{(t)} \neq h_1^{(t)}$, there exists a unique linear functional defined on $\mathscr{D}(\mathscr{L}^{(t)})$ such that the orthogonality relation*

$$\mathscr{L}^{(t)}\left[\frac{x^m \varphi_n^{(t)}(x)}{K_n^{(t)}(x)L_n(x)}\right] = h_n^{(t)}\delta_{m,n}, \quad n = 0, 1, 2, \dots, \quad m = 0, 1, 2, \dots, n,$$

holds, where $h_n^{(t)}$, $n = 2, 3, \dots$, are some nonzero constants.

In the rest of this thesis, we consider the *monic* $\mathrm{R_{II}}$ polynomials, i.e. the case where $\tilde{\alpha}_n^{(t)} = 1 + \beta_n^{(t)}$ holds for all $n = 0, 1, 2, \dots$. For general tridiagonal semi-infinite matrices of the form $\boldsymbol{B}^{(t)}$, if $\det(\boldsymbol{B}_n^{(t)}) \neq 0$ holds for all $n = 1, 2, 3, \dots$, then $\left\{\frac{\varphi_n^{(t)}(x)}{\det(\boldsymbol{B}_n^{(t)})}\right\}_{n=0}^{\infty}$ are the monic $\mathrm{R_{II}}$ polynomials. Therefore, the following argument is valid for such matrices.

The moment of the $\mathrm{R_{II}}$ linear functional $\mathscr{L}^{(t)}$ is introduced by

$$\mu_m^{k,l,t} := \mathscr{L}^{(t)}\left[\frac{x^m}{K_k^{(t)}(x)L_l(x)}\right], \quad k, l = 0, 1, 2, \dots, \quad m = 0, 1, \dots, k + l, \tag{5.2}$$

and its Hankel determinant by

$$\tau_0^{k,l,t} := 1, \quad \tau_n^{k,l,t} := |\mu_{i+j}^{k,l,t}|_{0 \le i,j \le n-1} = \begin{vmatrix} \mu_0^{k,l,t} & \mu_1^{k,l,t} & \dots & \mu_{n-1}^{k,l,t} \\ \mu_1^{k,l,t} & \mu_2^{k,l,t} & \dots & \mu_n^{k,l,t} \\ \vdots & \vdots & & \vdots \\ \mu_{n-1}^{k,l,t} & \mu_n^{k,l,t} & \dots & \mu_{2n-2}^{k,l,t} \end{vmatrix}, \quad n = 1, 2, 3, \dots.$$

Then, the determinant expression of the monic $\mathrm{R_{II}}$ polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^{\infty}$ is presented:

$$\varphi_n^{(t)}(x) = \frac{1}{\tau_n^{n,n,t}} \begin{vmatrix} \mu_0^{n,n,t} & \mu_1^{n,n,t} & \dots & \mu_{n-1}^{n,n,t} & \mu_n^{n,n,t} \\ \mu_1^{n,n,t} & \mu_2^{n,n,t} & \dots & \mu_n^{n,n,t} & \mu_{n+1}^{n,n,t} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{n-1}^{n,n,t} & \mu_n^{n,n,t} & \dots & \mu_{2n-2}^{n,n,t} & \mu_{2n-1}^{n,n,t} \\ 1 & x & \dots & x^{n-1} & x^n \end{vmatrix}, \quad n = 0, 1, 2, \dots. \tag{5.3}$$

5.2 Derivation of the $\mathrm{R_{II}}$ chain and its solutions

The discrete time evolution for the monic $\mathrm{R_{II}}$ polynomials is introduced by an analogue of the spectral transformations for monic orthogonal polynomials (2.6) and (2.7):

$$(x - s^{(t)})(1 + q_n^{(t)})\varphi_n^{(t+1)}(x) = \varphi_{n+1}^{(t)}(x) + q_n^{(t)}(x - \kappa_{t+n})\varphi_n^{(t)}(x), \tag{5.4a}$$

$$(1 + e_n^{(t)})\varphi_n^{(t)}(x) = \varphi_n^{(t+1)}(x) + e_n^{(t)}(x - \lambda_n)\varphi_{n-1}^{(t+1)}(x) \tag{5.4b}$$

for $n = 0, 1, 2, \dots$, where

$$q_n^{(t)} := -(s^{(t)} - \kappa_{t+n})\frac{\varphi_{n+1}^{(t)}(s^{(t)})}{\varphi_n^{(t)}(s^{(t)})}, \quad n = 0, 1, 2, \dots, \tag{5.5}$$

and $e_n^{(t)}$ is the variable determined by the compatibility condition:

$$\begin{aligned} 1 + \beta_n^{(t)} &= -q_n^{(t)} - e_n^{(t)}\frac{1 + q_n^{(t)}}{1 + q_{n-1}^{(t)}} + (1 + q_n^{(t)})(1 + e_n^{(t)}) \\ &= -q_n^{(t-1)}\frac{1 + e_{n+1}^{(t-1)}}{1 + e_n^{(t-1)}} - e_{n+1}^{(t-1)} + (1 + q_n^{(t-1)})(1 + e_{n+1}^{(t-1)}), \end{aligned} \tag{5.6a}$$

$$\begin{aligned} \alpha_n^{(t)} &= -\kappa_{t+n} q_n^{(t)} - \lambda_n e_n^{(t)}\frac{1 + q_n^{(t)}}{1 + q_{n-1}^{(t)}} + s^{(t)}(1 + q_n^{(t)})(1 + e_n^{(t)}) \\ &= -\kappa_{t+n-1} q_n^{(t-1)}\frac{1 + e_{n+1}^{(t-1)}}{1 + e_n^{(t-1)}} - \lambda_{n+1} e_{n+1}^{(t-1)} + s^{(t-1)}(1 + q_n^{(t-1)})(1 + e_{n+1}^{(t-1)}), \end{aligned} \tag{5.6b}$$

$$\beta_n^{(t)} = q_{n-1}^{(t)} e_n^{(t)}\frac{1 + q_n^{(t)}}{1 + q_{n-1}^{(t)}} = q_n^{(t-1)} e_n^{(t-1)}\frac{1 + e_{n+1}^{(t-1)}}{1 + e_n^{(t-1)}} \tag{5.6c}$$

with the boundary condition

$$e_0^{(t)} = 0 \quad \text{for all } t. \tag{5.6d}$$

This is the *monic type semi-infinite R_{II} chain*. Note that, since (5.6a) and (5.6c) are identical, there are the two independent equations that determine the time evolution of the two variables $q_n^{(t)}$ and $e_n^{(t)}$. One can readily verify that if $\{\varphi_n^{(t)}(x)\}_{n=0}^{\infty}$ are the monic $\mathrm{R_{II}}$ polynomials with respect to $\mathscr{L}^{(t)}$, then the polynomials $\{\varphi_n^{(t+1)}(x)\}_{n=0}^{\infty}$ defined by the spectral transformation (5.4a) are also the monic $\mathrm{R_{II}}$ polynomials, where the corresponding $\mathrm{R_{II}}$ linear functional is defined by

$$\mathscr{L}^{(t+1)}[R(x)] := \mathscr{L}^{(t)}\left[\frac{x - s^{(t)}}{x - \kappa_t} R(x)\right] \tag{5.7}$$

for all $R(x) \in \mathscr{D}(\mathscr{L}^{(t+1)})$.

Let us derive a solution to the monic type $\mathrm{R_{II}}$ chain. By the definition of the moment (5.2) and the time evolution of the linear functional (5.7), we obtain the relations

$$\mu_m^{k,l,t} = \mu_{m+1}^{k+1,l,t} - \kappa_{t+k}\mu_m^{k+1,l,t} = \mu_{m+1}^{k,l+1,t} - \lambda_{l+1}\mu_m^{k,l+1,t}, \tag{5.8a}$$

$$\mu_m^{k,l,t+1} = \mu_{m+1}^{k+1,l,t} - s^{(t)}\mu_m^{k+1,l,t}. \tag{5.8b}$$

The relation (5.8b), the determinant expression of the monic R_{II} polynomials (5.3), and the definition of the variable (5.5) lead to

$$q_n^{(t)} = (s^{(t)} - \kappa_{t+n})^{-1} \frac{\tau_n^{n,n,t} \tau_{n+1}^{n,n+1,t+1}}{\tau_n^{n-1,n,t+1} \tau_{n+1}^{n+1,n+1,t}}. \tag{5.9}$$

Next, the relation (5.8a) and the spectral transformation (5.4a) yield

$$1 + q_n^{(t)} = (\kappa_{t+n} - s^{(t)})^{-1} \frac{\varphi_{n+1}^{(t)}(\kappa_{t+n})}{\varphi_n^{(t+1)}(\kappa_{t+n})} = (s^{(t)} - \kappa_{t+n})^{-1} \frac{\tau_{n+1}^{n,n+1,t} \tau_n^{n,n,t+1}}{\tau_{n+1}^{n+1,n+1,t} \tau_n^{n-1,n,t+1}}.$$

Furthermore, the Jacobi identity for determinants [28, Section 2.6] proves the bilinear equation

$$\tau_n^{k-1,l-1,t} \tau_n^{k,l,t} - \tau_n^{k-1,l,t} \tau_n^{k,l-1,t} - \tau_{n-1}^{k-1,l-1,t} \tau_{n+1}^{k,l,t} = 0.$$

By using this bilinear equation and the three-term recurrence relation (5.1), we obtain

$$\beta_n^{(t)} = \frac{\mathscr{L}^{(t)} \left[\frac{x^{n+1} \varphi_{n+1}^{(t)}(x)}{K_{n+1}^{(t)}(x) L_{n+1}(x)} - \frac{x^n \varphi_n^{(t)}(x)}{K_n^{(t)}(x) L_n(x)} \right]}{\mathscr{L}^{(t)} \left[\frac{x^n \varphi_n^{(t)}(x)}{K_n^{(t)}(x) L_n(x)} - \frac{x^{n-1} \varphi_{n-1}^{(t)}(x)}{K_{n-1}^{(t)}(x) L_{n-1}(x)} \right]} = \frac{\tau_{n-1}^{n-1,n-1,t} \tau_{n+1}^{n,n+1,t} \tau_{n+1}^{n+1,n,t}}{\tau_n^{n-1,n,t} \tau_n^{n,n-1,t} \tau_{n+1}^{n+1,n+1,t}}.$$

Hence, from equation (5.6c) and these formulae, we find a solution

$$e_n^{(t)} = \frac{\beta_n^{(t)}}{q_{n-1}^{(t)}} \frac{1 + q_{n-1}^{(t)}}{1 + q_n^{(t)}} = (s^{(t)} - \kappa_{t+n}) \frac{\tau_{n-1}^{n-1,n-1,t+1} \tau_{n+1}^{n+1,n,t}}{\tau_n^{n,n-1,t} \tau_n^{n,n,t+1}}. \tag{5.10}$$

If the moments $\mu_m^{k,l,t}$ are arbitrary functions satisfying the relations (5.8), e.g.,

$$\mu_m^{k,l,t} = \int_\Omega \frac{x^m \prod_{j=0}^{t-1} (x - s^{(j)})}{K_{t+n}(x) L_n(x)} w(x) \, \mathrm{d}x,$$

then (5.9) and (5.10) give a solution to the monic type R_{II} chain (5.6) expressed by the Hankel determinant $\tau_n^{k,l,t}$.

The reason why the Hankel determinant appears can be explained from the point of view of the discrete two-dimensional Toda hierarchy [84]. Note that there is another determinant expression of the R_{II} polynomials and a solution to the R_{II} chain: the Casorati type determinant solution [52, 70].

5.3 Construction of a generalized eigenvalue algorithm for tridiagonal matrix pencils

In this section, we first derive the solution to the initial value problem and the convergence theorem for the monic type finite R_{II} chain. Next, using these results, we construct a generalized eigenvalue algorithm for tridiagonal matrix pencils based on the monic type finite R_{II} chain.

5.3.1 Solution to the initial value problem of the monic type finite $\mathrm{R_{II}}$ chain

Let us start with a pair of tridiagonal matrices of order N:

$$A^{(t)} = \begin{pmatrix} \alpha_0^{(t)} & \kappa_t & & & \\ \lambda_1\beta_1^{(t)} & \alpha_1^{(t)} & \kappa_{t+1} & & \\ & \lambda_2\beta_2^{(t)} & \ddots & \ddots & \\ & & \ddots & \ddots & \kappa_{t+N-2} \\ & & & \lambda_{N-1}\beta_{N-1}^{(t)} & \alpha_{N-1}^{(t)} \end{pmatrix}, \tag{5.11a}$$

$$B^{(t)} = \begin{pmatrix} 1 & 1 & & & \\ \beta_1^{(t)} & 1+\beta_1^{(t)} & 1 & & \\ & \beta_2^{(t)} & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \beta_{N-1}^{(t)} & 1+\beta_{N-1}^{(t)} \end{pmatrix}. \tag{5.11b}$$

The corresponding monic type finite $\mathrm{R_{II}}$ chain is

$$\begin{aligned} &\kappa_{t+n+1}q_n^{(t+1)} + \lambda_n e_n^{(t+1)}\frac{1+q_n^{(t+1)}}{1+q_{n-1}^{(t+1)}} - s^{(t+1)}(1+q_n^{(t+1)})(1+e_n^{(t+1)}) \\ &\qquad = \kappa_{t+n}q_n^{(t)}\frac{1+e_{n+1}^{(t)}}{1+e_n^{(t)}} + \lambda_{n+1}e_{n+1}^{(t)} - s^{(t)}(1+q_n^{(t)})(1+e_{n+1}^{(t)}), \end{aligned} \tag{5.12a}$$

$$q_{n-1}^{(t+1)}e_n^{(t+1)}\frac{1+q_n^{(t+1)}}{1+q_{n-1}^{(t+1)}} = q_n^{(t)}e_n^{(t)}\frac{1+e_{n+1}^{(t)}}{1+e_n^{(t)}}, \tag{5.12b}$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t. \tag{5.12c}$$

To derive the solution to the initial value problem for the monic type finite $\mathrm{R_{II}}$ chain (5.12), we consider the monic finite $\mathrm{R_{II}}$ polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^{\infty}$ defined by $\varphi_n^{(t)}(x) := \det(xB_n^{(t)} - A_n^{(t)})$. We should remark that $\varphi_N^{(t)}(x)$ is the characteristic polynomial of the tridiagonal matrix pencil $(A^{(t)}, B^{(t)})$; the zeros of the polynomial $\varphi_N^{(t)}(x)$ are the generalized eigenvalues of the matrix pencil $(A^{(t)}, B^{(t)})$, i.e., the solutions of the equation

$$A^{(t)}\boldsymbol{\Phi} = xB^{(t)}\boldsymbol{\Phi}, \quad x \in \mathbb{R}, \quad \boldsymbol{\Phi} \in \mathbb{R}^N - \{\mathbf{0}\}.$$

Let $\mathcal{D}(\mathcal{L}^{(t)})$ be a linear space spanned by the rational functions $\frac{x^m}{K_N^{(t)}(x)L_N(x)}$, $m = 0, 1, 2, \ldots$. For the monic finite $\mathrm{R_{II}}$ polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^{N}$ and any nonzero constant $H^{(t)}$, there exists a unique linear functional defined on $\mathcal{D}(\mathcal{L}^{(t)})$ such that the orthogonality relation

$$\mathcal{L}^{(t)}\left[\frac{x^m\varphi_n^{(t)}(x)}{K_n^{(t)}(x)L_n(x)}\right] = h_n^{(t)}\delta_{m,n}, \quad n = 0, 1, \ldots, N-1, \quad m = 0, 1, \ldots, n, \tag{5.13a}$$

and the terminating condition

$$\mathcal{L}^{(t)}\left[\frac{x^m\varphi_N^{(t)}(x)}{K_k^{(t)}(x)L_l(x)}\right] = 0, \qquad k, l = 0, 1, \ldots, N, \qquad m = 0, 1, 2, \ldots, \tag{5.13b}$$

hold, where the constants $h_0^{(t)}, h_1^{(t)}, \dots, h_{N-1}^{(t)}$ are given by solving the following linear equation

$$\begin{pmatrix} -1 & 1 & & & \\ \beta_1^{(t)} & -(1+\beta_1^{(t)}) & 1 & & \\ & \beta_2^{(t)} & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \beta_{N-1}^{(t)} & -(1+\beta_{N-1}^{(t)}) \end{pmatrix} \begin{pmatrix} h_0^{(t)} \\ h_1^{(t)} \\ \vdots \\ h_{N-1}^{(t)} \end{pmatrix} = \begin{pmatrix} -H^{(t)} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{aligned} h_0^{(t)} &= H^{(t)}(1+\beta_1^{(t)}+\beta_1^{(t)}\beta_2^{(t)}+\cdots+\beta_1^{(t)}\beta_2^{(t)}\dots\beta_{N-1}^{(t)}), \\ h_1^{(t)} &= H^{(t)}(\beta_1^{(t)}+\beta_1^{(t)}\beta_2^{(t)}+\cdots+\beta_1^{(t)}\beta_2^{(t)}\dots\beta_{N-1}^{(t)}), \\ &\vdots \\ h_{N-1}^{(t)} &= H^{(t)}\beta_1^{(t)}\beta_2^{(t)}\dots\beta_{N-1}^{(t)}. \end{aligned}$$

Note that, for the infinite dimensional case, there are two degrees of freedom: the choice of the two constants $h_0^{(t)}$ and $h_1^{(t)}$ (see Theorem 5.1). For the finite dimensional case, however, there is only one degree of freedom: the choice of the constant $H^{(t)}$. The cause of this is the terminating condition (5.13b).

To derive a realization of $\mathscr{L}^{(t)}$, we give a quadrature formula for the $\mathrm{R_{II}}$ linear functional. Suppose that all the zeros $x_0, x_1, \dots, x_{N-1}$ of the characteristic polynomial $\varphi_N^{(t)}(x)$ are simple.

Theorem 5.2 (The quadrature formula for the $\mathrm{R_{II}}$ linear functional)**.** *Let $x_0, x_1, \dots, x_{N-1}$ be the simple zeros of the characteristic polynomial $\varphi_N^{(t)}(x)$. For the linear functional $\mathscr{L}^{(t)}$ of the monic finite R_{II} polynomials $\{\varphi_n^{(t)}(x)\}_{n=0}^N$, there exist some constants $c_0^{(t)}, c_1^{(t)}, \dots, c_{N-1}^{(t)}$ such that*

$$\mathscr{L}^{(t)}[R(x)] = \sum_{i=0}^{N-1} c_i^{(t)} R(x_i) \tag{5.14}$$

holds for all $R(x) \in \mathscr{D}(\mathscr{L}^{(t)})$.

Proof. This proof is an analogue of the proof to the Gauss quadrature formula (Theorem 2.5). For the given rational function $R(x)$, consider the following interpolation rational function

$$\Lambda(x) \coloneqq \sum_{i=0}^{N-1} \ell_i^{(t)}(x) R(x_i),$$

where

$$\ell_i^{(t)}(x) \coloneqq \frac{\varphi_N^{(t)}(x) K_N^{(t)}(x_i) L_N(x_i)}{(x-x_i) K_N^{(t)}(x) L_N(x) {\varphi'}_N^{(t)}(x_i)}, \quad i = 0, 1, \dots, N-1.$$

It is readily shown that

$$\ell_i^{(t)}(x_j) = \delta_{i,j}, \quad i, j = 0, 1, \dots, N-1,$$

holds. Let

$$Q(x) \coloneqq R(x) - \Lambda(x).$$

Then, the numerator of $Q(x)$ is a polynomial that has zeros at $x_0, x_1, \ldots, x_{N-1}$. Since $R(x) \in \mathscr{D}(\mathscr{L}^{(t)})$, there exists a polynomial $P(x)$ such that

$$Q(x) = \frac{P(x)\varphi_N^{(t)}(x)}{K_N^{(t)}(x)L_N(x)}.$$

By the terminating condition (5.13b), we obtain

$$\begin{aligned}\mathscr{L}^{(t)}[R(x)] &= \mathscr{L}^{(t)}[\Lambda(x)] + \mathscr{L}^{(t)}[Q(x)] \\ &= \sum_{i=0}^{N-1} \mathscr{L}^{(t)}[\ell_i^{(t)}(x)]R(x_i) + \mathscr{L}^{(t)}\left[\frac{P(x)\varphi_N^{(t)}(x)}{K_N^{(t)}(x)L_N(x)}\right] \\ &= \sum_{i=0}^{N-1} \mathscr{L}^{(t)}[\ell_i^{(t)}(x)]R(x_i).\end{aligned}$$

Set $c_i^{(t)} := \mathscr{L}^{(t)}[\ell_i^{(t)}(x)]$, $i = 0, 1, \ldots, N-1$, then the proof is completed. ■

Zhedanov [91] derived a formula to calculate the constants $c_0^{(t)}, c_1^{(t)}, \ldots, c_{N-1}^{(t)}$. He used the second kind polynomials to derive it. Here, we give a direct calculation to check his result. From the quadrature formula (5.14), the moment is written as

$$\mu_m^{k,l,t} = \mathscr{L}^{(t)}\left[\frac{x^m}{K_k^{(t)}(x)L_l(x)}\right] = \sum_{i=0}^{N-1} \frac{c_i^{(t)}x_i^m}{K_k^{(t)}(x_i)L_l(x_i)}.$$

In the same manner as in Section 2.3, we thus obtain the following formulae for $j = 0, 1, \ldots, N-1$:

$$\varphi_{N-1}^{(t)}(x_j) = \frac{1}{\tau_{N-1}^{N-1,N-1,t}} \prod_{\substack{i=0\\ i\neq j}}^{N-1} \frac{c_i^{(t)}(x_j - x_i)}{K_{N-1}^{(t)}(x_i)L_{N-1}(x_i)} \prod_{\substack{0\le\nu_0<\nu_1\le N-1\\ \nu_0\neq j,\nu_1\neq j}} (x_{\nu_1} - x_{\nu_0})^2,$$

$$\varphi'^{(t)}_N(x_j) = \prod_{\substack{i=0\\ i\neq j}}^{N-1} (x_j - x_i),$$

and

$$\tau_N^{N-1,N-1,t} = \prod_{i=0}^{N-1} \frac{c_i^{(t)}}{K_{N-1}^{(t)}(x_i)L_{N-1}(x_i)} \prod_{\substack{0\le\nu_0<\nu_1\le N-1\\ \nu_0\neq j,\nu_1\neq j}} (x_{\nu_1} - x_{\nu_0})^2,$$

$$h_{N-1}^{(t)} = \frac{\tau_N^{N-1,N-1,t}}{\tau_{N-1}^{N-1,N-1,t}}.$$

Hence, we find the formula

$$c_i^{(t)} = \frac{h_{N-1}^{(t)}K_{N-1}^{(t)}(x_i)L_{N-1}(x_i)}{\varphi_{N-1}^{(t)}(x_i)\varphi'^{(t)}_N(x_i)}, \quad i = 0, 1, \ldots, N-1.$$

For the finite dimensional case, in the same manner as for the monic finite orthogonal polynomials (see Section 2.3), the characteristic polynomial is invariant under the time evolution:

$$\varphi_N^{(t+1)}(x) = \varphi_N^{(t)}(x).$$

From the results in Section 5.2, we can thus see that the solution to the initial value problem for the monic type finite $\mathrm{R_{II}}$ chain is given by

$$q_n^{(t)} = (s^{(t)} - \kappa_{t+n})^{-1} \frac{\tau_n^{n,n,t} \tau_{n+1}^{n,n+1,t+1}}{\tau_n^{n-1,n,t+1} \tau_{n+1}^{n+1,n+1,t}}, \quad e_n^{(t)} = (s^{(t)} - \kappa_{t+n}) \frac{\tau_{n-1}^{n-1,n-1,t+1} \tau_{n+1}^{n+1,n,t}}{\tau_n^{n,n-1,t} \tau_n^{n,n,t+1}},$$

where, because the moment is concretely given by

$$\mu_m^{k,l,t} = \sum_{i=0}^{N-1} \frac{c_i^{(0)} x_i^m \prod_{j=0}^{t-1} (x_i - s^{(j)})}{K_{t+k}(x_i) L_l(x_i)},$$

the expanded form of the Hankel determinant is

$$\tau_n^{k,l,t} = \sum_{0 \le r_0 < r_1 < \cdots < r_{n-1} \le N-1} \left(\prod_{i=0}^{n-1} \frac{c_{r_i}^{(0)} \prod_{j=0}^{t-1} (x_{r_i} - s^{(j)})}{K_{t+k}(x_{r_i}) L_l(x_{r_i})} \prod_{0 \le \nu_0 < \nu_1 \le n-1} (x_{r_{\nu_1}} - x_{r_{\nu_0}})^2 \right).$$

5.3.2 Convergence theorem and matrix form

The solution derived above yields the following theorem.

Theorem 5.3 (Convergence theorem for the monic type finite $\mathrm{R_{II}}$ chain)**.** *Suppose that all the generalized eigenvalues $x_0, x_1, \dots, x_{N-1}$ of the initial tridiagonal matrix pencil $(A^{(0)}, B^{(0)})$ are real, simple and arranged in descending order as $x_0 > x_1 > \cdots > x_{N-1}$. Choose the parameters $s^{(t)}$ and κ_{t+N-1} as $x_{N-1} > s^{(t)}$ and $x_{N-1} \gg \kappa_{t+N-1}$ for all t, respectively. Then, we have the asymptotics of the variables for $t \gg 1$:*

$$\begin{aligned} q_n^{(t)} &= \frac{x_n - s^{(t)}}{s^{(t)} - \kappa_{t+n}} + O\left(\max \left\{ \frac{\prod_{j=0}^{t} (x_n - s^{(j)})}{\prod_{j=0}^{t-1} (x_{n-1} - s^{(j)})} \frac{\prod_{j=0}^{t+n-1} (x_{n-1} - \kappa_j)}{\prod_{j=0}^{t+n-1} (x_n - \kappa_j)}, \right.\right. \\ &\qquad \left.\left. \frac{\prod_{j=0}^{t} (x_{n+1} - s^{(j)})}{\prod_{j=0}^{t-1} (x_n - s^{(j)})} \frac{\prod_{j=0}^{t+n-1} (x_n - \kappa_j)}{\prod_{j=0}^{t+n-1} (x_{n+1} - \kappa_j)} \right\} \right), \\ e_n^{(t)} &= O\left(\frac{\prod_{j=0}^{t-1} (x_n - s^{(j)})}{\prod_{j=0}^{t} (x_{n-1} - s^{(j)})} \frac{\prod_{j=0}^{t+n-1} (x_{n-1} - \kappa_j)}{\prod_{j=0}^{t+n} (x_n - \kappa_j)} \right). \end{aligned}$$

Hence, the variables $q_n^{(t)}$ and $e_n^{(t)}$ converge to $\frac{x_n - s^{(t)}}{s^{(t)} - \kappa_{t+n}}$ and 0 as $t \to +\infty$, respectively.

This theorem implies that, from (5.6), the elements $\alpha_n^{(t)}$ and $\beta_n^{(t)}$ of the tridiagonal matrices $A^{(t)}$ and $B^{(t)}$ converge to x_n and 0 as $t \to +\infty$, respectively. Furthermore, we can see that the parameters $s^{(t)}$ and κ_{t+n} determine the convergence speed; the parameter $s^{(t)}$ works as the origin shift, which is the same as for the dqds algorithm (see Section 2.3).

Next, we discuss the matrix form of the monic type finite $\mathrm{R_{II}}$ chain. Introduce the rational functions defined by the following three-term recurrence relation:

$$\begin{aligned} &\Phi_{-1}^{(t)}(x) := 0, \quad \Phi_0^{(t)}(x) := 1, \\ &(x - \kappa_{t+n}) \Phi_{n+1}^{(t)}(x) := -\left((1 + \beta_n^{(t)}) x - \alpha_n^{(t)} \right) \Phi_n^{(t)}(x) - \beta_n^{(t)} (x - \lambda_n) \Phi_{n-1}^{(t)}(x), \\ &\qquad\qquad n = 0, 1, \dots, N-1. \end{aligned} \tag{5.15}$$

By comparing to the three-term recurrence relation (5.1), the relation

$$\Phi_n^{(t)}(x) = \frac{\varphi_n^{(t)}(x)}{K_n^{(t)}(x)}, \quad n = 0, 1, \dots, N,$$

is verified. Let

$$\boldsymbol{\Phi}^{(t)}(x) := \begin{pmatrix} \Phi_0^{(t)}(x) \\ \Phi_1^{(t)}(x) \\ \vdots \\ \Phi_{N-1}^{(t)}(x) \end{pmatrix}, \quad \boldsymbol{\Phi}_N^{(t)}(x) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi_N^{(t)}(x) \end{pmatrix}.$$

Then, the three-term recurrence relation (5.15) is rewritten as

$$A^{(t)}\boldsymbol{\Phi}^{(t)}(x) + \kappa_{t+N}\boldsymbol{\Phi}_N^{(t)}(x) = x\left(B^{(t)}\boldsymbol{\Phi}^{(t)}(x) + \boldsymbol{\Phi}_N^{(t)}(x)\right). \tag{5.16a}$$

Furthermore, let $L_{\mathrm{A}}^{(t)}$, $L_{\mathrm{B}}^{(t)}$, and $R^{(t)}$ be bidiagonal matrices:

$$L_{\mathrm{A}}^{(t)} := \begin{pmatrix} \kappa_t & & & & \\ -\lambda_1 e_1^{(t)} & \kappa_{t+1} & & & \\ & -\lambda_2 e_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\lambda_{N-1} e_{N-1}^{(t)} & \kappa_{t+N-1} \end{pmatrix},$$

$$L_{\mathrm{B}}^{(t)} := \begin{pmatrix} 1 & & & & \\ -e_1^{(t)} & 1 & & & \\ & -e_2^{(t)} & \ddots & & \\ & & \ddots & \ddots & \\ & & & -e_{N-1}^{(t)} & 1 \end{pmatrix}, \quad R^{(t)} := \begin{pmatrix} q_0^{(t)} & -1 & & & \\ & q_1^{(t)} & -1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & -1 \\ & & & & q_{N-1}^{(t)} \end{pmatrix},$$

and $D_{\mathrm{q}}^{(t)}$, $D_{\mathrm{e}}^{(t)}$ and $\hat{D}_{\mathrm{e}}^{(t)}$ be diagonal matrices:

$$\begin{aligned} D_{\mathrm{q}}^{(t)} &:= \operatorname{diag}\left(1 + q_0^{(t)}, 1 + q_1^{(t)}, \dots, 1 + q_{N-1}^{(t)}\right), \\ D_{\mathrm{e}}^{(t)} &:= \operatorname{diag}\left(1, 1 + e_1^{(t)}, \dots, 1 + e_{N-1}^{(t)}\right), \quad \hat{D}_{\mathrm{e}}^{(t)} := \operatorname{diag}\left(1 + e_1^{(t)}, \dots, 1 + e_{N-1}^{(t)}, 1\right). \end{aligned}$$

Then, the spectral transformations (5.4) are written in terms of the rational functions $\{\Phi_n^{(t)}(x)\}_{n=0}^{N}$ as

$$(x - s^{(t)})D_{\mathrm{q}}^{(t)}\boldsymbol{\Phi}^{(t+1)}(x) = (x - \kappa_t)\left(R^{(t)}\boldsymbol{\Phi}^{(t)}(x) - \boldsymbol{\Phi}_N^{(t)}(x)\right), \tag{5.16b}$$

$$D_{\mathrm{e}}^{(t)}\boldsymbol{\Phi}^{(t)}(x) = (x - \kappa_t)^{-1}\left(xL_{\mathrm{B}}^{(t)} - L_{\mathrm{A}}^{(t)}\right)\boldsymbol{\Phi}^{(t+1)}(x). \tag{5.16c}$$

Equations (5.16) yield

$$\begin{aligned} & x\left(B^{(t+1)}\boldsymbol{\Phi}^{(t+1)}(x) + \boldsymbol{\Phi}_N^{(t+1)}(x)\right) - A^{(t+1)}\boldsymbol{\Phi}^{(t+1)}(x) - \kappa_{t+N}\boldsymbol{\Phi}_N^{(t+1)}(x) \\ =& x\left(\left(-D_{\mathrm{q}}^{(t+1)}L_{\mathrm{B}}^{(t+1)}(D_{\mathrm{q}}^{(t+1)})^{-1}R^{(t+1)} + D_{\mathrm{q}}^{(t+1)}D_{\mathrm{e}}^{(t+1)}\right)\boldsymbol{\Phi}^{(t+1)}(x) + \boldsymbol{\Phi}_N^{(t+1)}(x)\right) \\ & - \left(-D_{\mathrm{q}}^{(t+1)}L_{\mathrm{A}}^{(t+1)}(D_{\mathrm{q}}^{(t+1)})^{-1}R^{(t+1)} + s^{(t+1)}D_{\mathrm{q}}^{(t+1)}D_{\mathrm{e}}^{(t+1)}\right)\boldsymbol{\Phi}^{(t+1)}(x) - \kappa_{t+N}\boldsymbol{\Phi}_N^{(t+1)}(x) \end{aligned}$$

$$
\begin{aligned}
&= x\left(\left(-\hat{D}_{\mathrm{e}}^{(t)} R^{(t)} (D_{\mathrm{e}}^{(t)})^{-1} L_{\mathrm{B}}^{(t)} + D_{\mathrm{q}}^{(t)} \hat{D}_{\mathrm{e}}^{(t)}\right) \boldsymbol{\Phi}^{(t+1)}(x) + \boldsymbol{\Phi}_N^{(t+1)}(x)\right) \\
&\quad - \left(-\hat{D}_{\mathrm{e}}^{(t)} R^{(t)} (D_{\mathrm{e}}^{(t)})^{-1} L_{\mathrm{A}}^{(t)} + s^{(t)} D_{\mathrm{q}}^{(t)} \hat{D}_{\mathrm{e}}^{(t)}\right) \boldsymbol{\Phi}^{(t+1)}(x) - \kappa_{t+N} \boldsymbol{\Phi}_N^{(t+1)}(x) \\
&= \mathbf{0}.
\end{aligned}
$$

Hence, the compatibility condition for (5.16), i.e. the matrix form of the monic type finite R_{II} chain, is given by

$$
\begin{aligned}
A^{(t+1)} &= -D_{\mathrm{q}}^{(t+1)} L_{\mathrm{A}}^{(t+1)} (D_{\mathrm{q}}^{(t+1)})^{-1} R^{(t+1)} + s^{(t+1)} D_{\mathrm{q}}^{(t+1)} D_{\mathrm{e}}^{(t+1)} \\
&= -\hat{D}_{\mathrm{e}}^{(t)} R^{(t)} (D_{\mathrm{e}}^{(t)})^{-1} L_{\mathrm{A}}^{(t)} + s^{(t)} D_{\mathrm{q}}^{(t)} \hat{D}_{\mathrm{e}}^{(t)}, \\
B^{(t+1)} &= -D_{\mathrm{q}}^{(t+1)} L_{\mathrm{B}}^{(t+1)} (D_{\mathrm{q}}^{(t+1)})^{-1} R^{(t+1)} + D_{\mathrm{q}}^{(t+1)} D_{\mathrm{e}}^{(t+1)} \\
&= -\hat{D}_{\mathrm{e}}^{(t)} R^{(t)} (D_{\mathrm{e}}^{(t)})^{-1} L_{\mathrm{B}}^{(t)} + D_{\mathrm{q}}^{(t)} \hat{D}_{\mathrm{e}}^{(t)}.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
A^{(t+1)} &= \hat{D}_{\mathrm{e}}^{(t)} R^{(t)} (D_{\mathrm{q}}^{(t)} D_{\mathrm{e}}^{(t)})^{-1} A^{(t)} (R^{(t)})^{-1} D_{\mathrm{q}}^{(t)}, \\
B^{(t+1)} &= \hat{D}_{\mathrm{e}}^{(t)} R^{(t)} (D_{\mathrm{q}}^{(t)} D_{\mathrm{e}}^{(t)})^{-1} B^{(t)} (R^{(t)})^{-1} D_{\mathrm{q}}^{(t)},
\end{aligned}
$$

and

$$
xB^{(t+1)} - A^{(t+1)} = \hat{D}_{\mathrm{e}}^{(t)} R^{(t)} (D_{\mathrm{q}}^{(t)} D_{\mathrm{e}}^{(t)})^{-1} \left(xB^{(t)} - A^{(t)}\right) (R^{(t)})^{-1} D_{\mathrm{q}}^{(t)}.
$$

The last equation implies that all the generalized eigenvalues of the tridiagonal matrix pencil $(A^{(t)}, B^{(t)})$ are conserved under the time evolution.

5.3.3 Generalized eigenvalue algorithm

In the previous subsection, we have presented the convergence theorem for the monic type finite R_{II} chain (Theorem 5.3). This theorem allows us to design a generalized eigenvalue algorithm for tridiagonal matrix pencils.

Consider a pair of tridiagonal matrices of order N as input:

$$
A = \begin{pmatrix}
a_{0,0} & a_{0,1} & & & \\
a_{1,0} & a_{1,1} & a_{1,2} & & \\
 & a_{2,1} & \ddots & \ddots & \\
 & & \ddots & \ddots & a_{N-2,N-1} \\
 & & & a_{N-1,N-2} & a_{N-1,N-1}
\end{pmatrix}, \tag{5.17a}
$$

$$
B = \begin{pmatrix}
b_{0,0} & b_{0,1} & & & \\
b_{1,0} & b_{1,1} & b_{1,2} & & \\
 & b_{2,1} & \ddots & \ddots & \\
 & & \ddots & \ddots & b_{N-2,N-1} \\
 & & & b_{N-1,N-2} & b_{N-1,N-1}
\end{pmatrix}. \tag{5.17b}
$$

Suppose that all the subdiagonal elements $b_{0,1}, b_{1,2}, \dots, b_{N-2,N-1}$ and $b_{1,0}, b_{2,0}, \dots, b_{N-1,N-2}$ of the matrix B are nonzero, and all the leading principal minors of the matrix B are nonzero. Then, the transformation

$$
A^{(0)} \coloneqq V_1 U A U^{-1} V_2, \quad B^{(0)} \coloneqq V_1 U B U^{-1} V_2
$$

gives the initial matrix pencil of the form (5.11) for the monic type finite $\mathrm{R_{II}}$ chain, where

$$
\begin{aligned}
U &:= \mathrm{diag}(1, b_{0,1}, b_{0,1}b_{1,2}, \dots, b_{0,1}b_{1,2}\dots b_{N-2,N-1}),\\
V_1 &:= \mathrm{diag}\left((\det \boldsymbol{B}_1)^{-1}, (\det \boldsymbol{B}_2)^{-1}, \dots, (\det \boldsymbol{B}_N)^{-1}\right),\\
V_2 &:= \mathrm{diag}(1, \det \boldsymbol{B}_1, \det \boldsymbol{B}_2, \dots, \det \boldsymbol{B}_{N-1}),
\end{aligned}
$$

and $\boldsymbol{B}_n$ is the n-th order leading principal submatrix of the matrix $\boldsymbol{B}$. Namely, the elements of $\boldsymbol{A}^{(0)}$ and $\boldsymbol{B}^{(0)}$ are computed by

$$
\alpha_n^{(0)} := a_{n,n}\frac{\det \boldsymbol{B}_n}{\det \boldsymbol{B}_{n+1}}, \quad \beta_n^{(0)} := b_{n-1,n}b_{n,n-1}\frac{\det \boldsymbol{B}_{n-1}}{\det \boldsymbol{B}_{n+1}}, \quad \kappa_n := \frac{a_{n,n+1}}{b_{n,n+1}}, \quad \lambda_n := \frac{a_{n,n-1}}{b_{n,n-1}}. \tag{5.18}
$$

Note that, if n is large, an overflow may occur when one computes $\det \boldsymbol{B}_n$ directly. The values $\frac{\det \boldsymbol{B}_n}{\det \boldsymbol{B}_{n+1}}$ and $\frac{\det \boldsymbol{B}_{n-1}}{\det \boldsymbol{B}_{n+1}}$ should be computed by the LU decomposition. Next, by the relation (5.6), "decompose" the matrix pencil $(\boldsymbol{A}^{(0)}, \boldsymbol{B}^{(0)})$ to the variables of the monic type finite $\mathrm{R_{II}}$ chain:

$$
e_0^{(0)} := 0, \quad e_N^{(0)} := 0, \tag{5.19a}
$$

$$
\tilde{e}_n^{(0)} := \frac{\beta_n^{(0)}}{q_{n-1}^{(0)}}, \quad e_n^{(0)} := \tilde{e}_n^{(0)}\frac{1 + q_{n-1}^{(0)}}{1 + q_n^{(0)}}, \qquad n = 1, 2, \dots, N-1, \tag{5.19b}
$$

$$
q_0^{(0)} := \frac{\alpha_0^{(0)} - s^{(0)}}{s^{(0)} - \kappa_0}, \tag{5.19c}
$$

$$
q_n^{(0)} := \frac{\alpha_n^{(0)} - s^{(0)}(1 + \beta_n^{(0)}) - (s^{(0)} - \lambda_n)\tilde{e}_n^{(0)}}{s^{(0)} - \kappa_n}, \qquad n = 1, 2, \dots, N-1. \tag{5.19d}
$$

Note that the initial matrix pencil $(\boldsymbol{A}^{(0)}, \boldsymbol{B}^{(0)})$ does not fix the values of the parameters $s^{(0)}$ and κ_{N-1}. We must choose the parameters $s^{(0)}$ and κ_{N-1} appropriately. We will discuss how to choose the parameters in the end of this subsection. After that, compute the time evolution of the monic type finite $\mathrm{R_{II}}$ chain by using (5.12) iteratively, i.e. for each $t \geq 0$, compute

$$
e_0^{(t+1)} := 0, \quad e_N^{(t+1)} := 0, \tag{5.20a}
$$

$$
e_n^{(t+1)} := e_n^{(t)}\frac{q_n^{(t)}}{q_{n-1}^{(t+1)}}\frac{1 + q_{n-1}^{(t+1)}}{1 + q_n^{(t+1)}}\frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}}, \quad n = 1, 2, \dots, N-1, \tag{5.20b}
$$

$$
\tag{5.20c}
$$

$$
\begin{aligned}
q_n^{(t+1)} := {}&(s^{(t+1)} - \kappa_{t+n+1})^{-1}\\
&\times \Bigg((s^{(t+1)} - \kappa_{t+n})q_n^{(t)}\frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}} - (s^{(t+1)} - \lambda_n)e_n^{(t+1)}\frac{1 + q_n^{(t+1)}}{1 + q_{n-1}^{(t+1)}}\\
&\quad + (s^{(t+1)} - \lambda_{n+1})e_{n+1}^{(t)} - (s^{(t+1)} - s^{(t)})(1 + q_n^{(t)})(1 + e_{n+1}^{(t)})\Bigg),\\
&\qquad n = 0, 1, \dots, N-1.
\end{aligned} \tag{5.20d}
$$

Here, we also have to choose the parameters $s^{(t+1)}$ and κ_{t+N} for computing the above recurrence equations. From the results in the previous subsection, we can see that if the absolute values of all the subdiagonal elements $\lambda_n\beta_n^{(t)}$ and $\beta_n^{(t)}$ of the matrix pencil $(\boldsymbol{A}^{(t)}, \boldsymbol{B}^{(t)})$ become sufficiently small at a time t, then the values $(s^{(t)} - \kappa_{t+n})q_n^{(t)} + s^{(t)}$ give the generalized eigenvalues of the initial tridiagonal matrix pencil $(\boldsymbol{A}, \boldsymbol{B})$. In general, however, equation (5.20d) requires subtraction operations, which may degrade the accuracy by the loss of significant digits. A subtraction-free form of the monic type finite $\mathrm{R_{II}}$ chain may resolve the problem.

Let us introduce the auxiliary variable

$$d_n^{(t+1)} = \frac{(s^{(t+1)} - \kappa_{t+n+1})q_n^{(t+1)} - (s^{(t+1)} - \lambda_{n+1})e_{n+1}^{(t)}}{1 + e_{n+1}^{(t)}}, \quad n = 0, 1, \dots, N-1.$$

This is an analogue of the auxiliary variable (2.24) introduced in the dqds algorithm. Then, the subtraction-free form is derived as

$$d_0^{(t+1)} := (s^{(t)} - \kappa_t)q_0^{(t)} - (s^{(t+1)} - s^{(t)}), \tag{5.21a}$$

$$d_n^{(t+1)} := d_{n-1}^{(t+1)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} - (s^{(t+1)} - s^{(t)})(1 + q_n^{(t)}), \quad n = 1, 2, \dots, N-1, \tag{5.21b}$$

$$q_n^{(t+1)} := \frac{(s^{(t+1)} - \lambda_{n+1})e_{n+1}^{(t)} + d_n^{(t+1)}(1 + e_{n+1}^{(t)})}{s^{(t+1)} - \kappa_{t+n+1}}, \quad n = 0, 1, \dots, N-1, \tag{5.21c}$$

$$e_n^{(t+1)} := e_n^{(t)} \frac{q_n^{(t)}}{q_{n-1}^{(t+1)}} \frac{1 + q_{n-1}^{(t+1)}}{1 + q_n^{(t+1)}} \frac{1 + e_{n+1}^{(t)}}{1 + e_n^{(t)}}, \quad n = 1, 2, \dots, N-1, \tag{5.21d}$$

$$e_0^{(t)} = e_N^{(t)} = 0 \quad \text{for all } t. \tag{5.21e}$$

From the spectral transformations (5.4), we have

$$-(1 + e_{n+1}^{(t-1)})\varphi_{n+1}^{(t-1)}(s^{(t)}) = \left((s^{(t)} - \kappa_{t+n})q_n^{(t)} - (s^{(t)} - \lambda_{n+1})e_{n+1}^{(t-1)}\right)\varphi_n^{(t)}(s^{(t)}),$$

which implies

$$d_n^{(t)} = -\frac{\varphi_{n+1}^{(t-1)}(s^{(t)})}{\varphi_n^{(t)}(s^{(t)})} = \frac{\tau_n^{n,n,t}\sigma_{n+1}^{n,n+1,t}}{\tau_{n+1}^{n+1,n+1,t-1}\tau_n^{n-1,n,t+1}},$$

where

$$\sigma_n^{k,l,t} := |\mu_{i+j+1}^{k+1,l,t-1} - s^{(t)}\mu_{i+j}^{k+1,l,t-1}|_{0\le i,j\le n-1}.$$

In addition, we already have the expression of $q_n^{(t)}$ (5.5). Hence, we obtain a sufficient condition for computing the recurrence equation (5.21) without subtraction operations except the shift terms in (5.21b): for all n and t,

$$\beta_n^{(0)} > 0, \tag{5.22a}$$

$$(-1)^n\varphi_n^{(t)}(s^{(t)}) = \det(A_n^{(t)} - s^{(t)}B_n^{(t)}) > 0, \tag{5.22b}$$

$$(-1)^n\varphi_n^{(t)}(s^{(t+1)}) = \det(A_n^{(t)} - s^{(t+1)}B_n^{(t)}) > 0, \tag{5.22c}$$

$$s^{(t)} > \kappa_{t+n}, \quad s^{(t)} > \lambda_n. \tag{5.22d}$$

By (5.18), if the input tridiagonal matrix B is a real symmetric positive (or negative) definite matrix, then the condition (5.22a) is satisfied. Furthermore, assume that the generalized eigenvalues $x_0, x_1, \dots, x_{N-1}$ of the input tridiagonal matrix pencil (A, B) are all real and simple, the matrix A is a real matrix and the conditions $\beta_n^{(t)} > 0$ and $\kappa_{t+n-1} = \lambda_n$ are satisfied for $n = 1, 2, \dots, N-1$ at some time t. Then, it is shown that if the parameter $s^{(t)}$ is chosen as $s^{(t)} < \min\{x_0, x_1, \dots, x_{N-1}\}$, the condition (5.22b) is satisfied. The condition (5.22c) is also satisfied with $s^{(t+1)} < \min\{x_0, x_1, \ldots, x_{N-1}\}$. From Theorem 5.3, if $s^{(t)}$ is chosen as close as possible to $\min\{x_0, x_1, \ldots, x_{N-1}\}$ under the conditions (5.22), the convergence speed is accelerated.

By summarizing this section, Algorithm 3 is proposed as a new generalized eigenvalue algorithm for tridiagonal matrix pencils based on the monic type finite $\mathrm{R_{II}}$ chain.

Algorithm 3 The proposed generalized eigenvalue algorithm based on the monic type finite $\mathrm{R_{II}}$ chain

function GEVRII(A, B) ▷ A and B are tridiagonal matrices of the form (5.17)
  Compute $\{\alpha_n^{(0)}\}_{n=0}^{N-1}$, $\{\beta_n^{(0)}\}_{n=1}^{N-1}$, $\{\kappa_n\}_{n=0}^{N-2}$, and $\{\lambda_n\}_{n=1}^{N-1}$ by (5.18)
  Set the parameters $s^{(0)}$ and κ_{N-1} appropriately ▷ See Theorem 5.3 and the condition (5.22)
  Compute $\{q_n^{(0)}\}_{n=0}^{N-1}$ and $\{e_n^{(0)}\}_{n=0}^{N}$ by (5.19)
  $t := 0$
  repeat
    Set the parameters $s^{(t+1)}$ and κ_{t+N} appropriately ▷ See Theorem 5.3 and the condition (5.22)
    Compute $\{q_n^{(t+1)}\}_{n=0}^{N-1}$ and $\{e_n^{(t+1)}\}_{n=0}^{N}$ by (5.21)
    $t := t + 1$
    for $n = 1, 2, \ldots, N-1$ **do**
      $\beta_n^{(t)} := q_{n-1}^{(t)} e_n^{(t)} \frac{1+q_n^{(t)}}{1+q_{n-1}^{(t)}}$
    end for
  until the absolute values of $\beta_n^{(t)}$ and $\lambda_n \beta_n^{(t)}$ are sufficiently small for all $n = 1, 2, \ldots, N-1$
  return $\{(s^{(t)} - \kappa_{t+n}) q_n^{(t)} + s^{(t)}\}_{n=0}^{N-1}$
end function

5.4 Numerical examples

We give numerical examples. To construct test problems with known generalized eigenvalues, let us consider the monic finite orthogonal polynomials $\{p_n(x)\}_{n=0}^{N}$ defined by

$$p_{n+1}(x) := \left(x - \frac{N-1}{2}\right) p_n(x) - \frac{n(N-n)}{4} p_{n-1}(x), \quad n = 0, 1, \ldots, N-1,$$

with $p_{-1}(x) := 0$ and $p_0(x) := 1$. The polynomials $\{p_n(x)\}_{n=0}^{N}$ are the monic Krawtchouk polynomials with a special parameter and it is well known that the Krawtchouk polynomials are orthogonal on $x = 0, 1, \ldots, N-1$ with respect to the binomial distribution [40]. This means that the tridiagonal matrix of order N

$$\tilde{K}_N := \begin{pmatrix} (N-1)/2 & 1 & & & & \\ (N-1)/4 & (N-1)/2 & 1 & & & \\ & 2(N-2)/4 & (N-1)/2 & 1 & & \\ & & 3(N-3)/4 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & (N-1)/4 & (N-1)/2 \end{pmatrix} \in \mathbb{R}^{N \times N}$$

has the eigenvalues $0, 1, \dots, N-1$. The symmetric tridiagonal matrix

$$K_N := \begin{pmatrix} (N-1)/2 & \sqrt{(N-1)/4} & & & \\ \sqrt{(N-1)/4} & (N-1)/2 & \sqrt{2(N-2)/4} & & \\ & \sqrt{2(N-2)/4} & \ddots & \ddots & \\ & & \ddots & \ddots & \sqrt{(N-1)/4} \\ & & & \sqrt{(N-1)/4} & (N-1)/2 \end{pmatrix} \in \mathbb{R}^{N\times N}$$

is similar to $\tilde{K}_N$. Hence, it is readily shown that the tridiagonal matrix pencil $(K_N + 2I_N, K_N + I_N)$ has the generalized eigenvalues $(n+1)/n$, $n = 1, 2, \dots, N$.

The following experiments were run on a Linux PC with kernel 3.7.4 and gcc 4.7.2 on Intel Core i5 760 2.80 GHz CPU and 4 GB memory. All the computations were performed in double precision and the stopping criterion (line 13 in Algorithm 3) was $|\beta_n^{(t)}| < 10^{-20}$ and $|\lambda_n \beta_n^{(t)}| < 10^{-20}$ for all $n = 1, 2, \dots, N-1$.

Example 5.4. The first example is the case of $N = 5$:

$$K_5 = \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & \sqrt{3/2} & & \\ & \sqrt{3/2} & 2 & \sqrt{3/2} & \\ & & \sqrt{3/2} & 2 & 1 \\ & & & 1 & 2 \end{pmatrix}.$$

The generalized eigenvalues of the matrix pencil $(K_5 + 2I_5, K_5 + I_5)$ are 2, 3/2, 4/3, 5/4, and 6/5. By this example, we will observe the behaviour of the variables of the monic type finite $\mathrm{R_{II}}$ chain and confirm that the proposed algorithm computes the generalized eigenvalues of a given matrix pencil and its convergence speed depends on the parameters $s^{(t)}$ and κ_{t+n}.

Figure 5.1 shows the result with the parameters $s^{(t)} = 1.01$ for all $t \geq 0$ and $\kappa_n = 1$ for all $n \geq 4$, where $q_n^{(t)}$ and $e_n^{(t)}$ are the variables of the monic type $\mathrm{R_{II}}$ chain, $\alpha_n^{(t)}$ are the diagonal elements of $A^{(t)}$, and $\beta_n^{(t)}$ are the subdiagonal elements of $B^{(t)}$ (see equations (5.6b) and (5.6c)). We can confirm that $\alpha_n^{(t)}$ and $\beta_n^{(t)}$ converge linearly to the eigenvalues and zero, respectively. Since the shift parameter $s^{(t)}$ is not so close to the minimal eigenvalue $6/5 = 1.2$, the stopping criterion is satisfied at $t = 4605$.

Table 5.1: The eigenvalues computed by Algorithm 3. The parameters are $s^{(t)} = 1.19$ and $\kappa_n = -10000$ for all $t \geq 0$ and $n \geq 4$.

Computed eigenvalues	True eigenvalues
1.9999999999999998	2.0000000000000000
1.4999999999999991	1.5000000000000000
1.3333333333333335	1.3333333333333333
1.2500000000000000	1.2500000000000000
1.2000000000000000	1.2000000000000000

Figure 5.2 shows the result with more suitable parameters: $s^{(t)} = 1.19$ for all $t \geq 0$ and $\kappa_n = -10000$ for all $n \geq 4$. The convergence speed is much faster than the former example; the stopping criterion is satisfied at $t = 48$. Table 5.1 shows the computed eigenvalues.

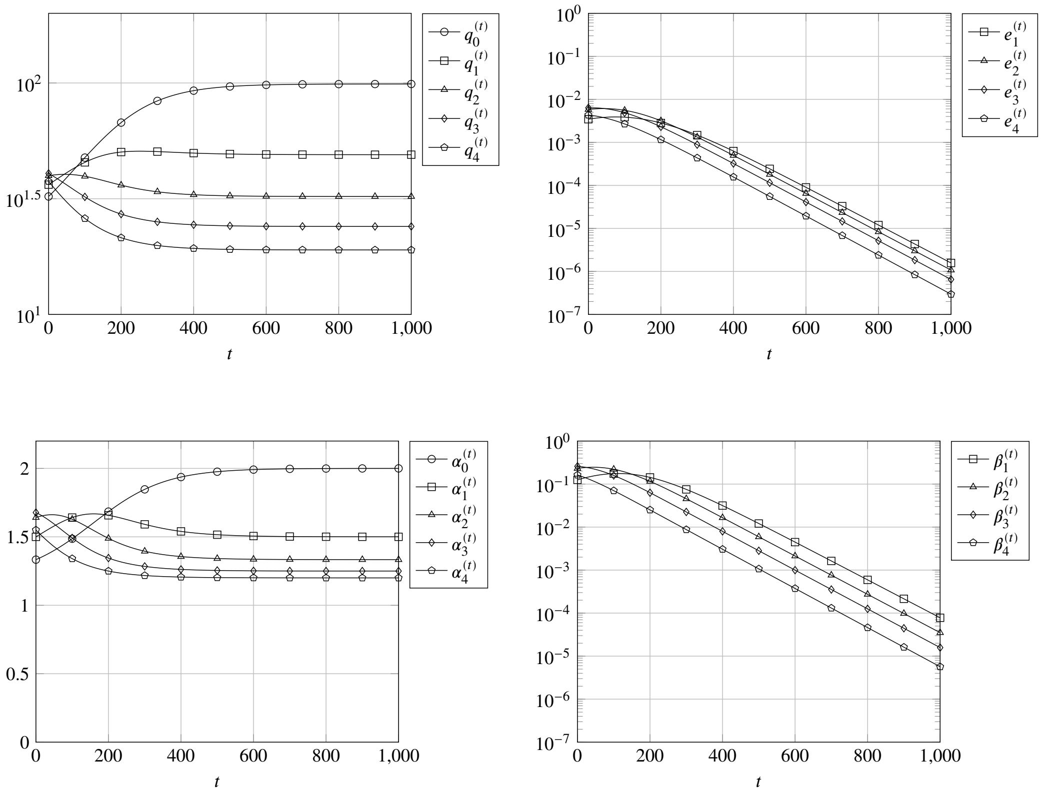


Figure 5.1: The behaviour of the variables of the monic type finite $\mathrm{R_{II}}$ chain for the input tridiagonal matrix pencil $(K_5 + 2I_5, K_5 + I_5)$ with the parameters $s^{(t)} = 1.01$ for all $t \geq 0$ and $\kappa_n = 1$ for all $n \geq 4$.

Example 5.5. Next, the test cases for $N = 512$, 1024, 2048, 4096, 8192 were computed by two methods. By these examples, we will compare the computation time and the accuracy of the proposed algorithm with a routine called DSYGV in LAPACK 3.4.2 [42]. DSYGV computes the generalized eigenvalues of a given matrix pencil (A, B) in double precision, where A is real symmetric and B is real symmetric and positive definite. Internally, DSYGV computes the Cholesky factorization $B = LL^{\mathrm{T}}$, where L is a lower triangular matrix, transforms the generalized eigenvalue problem $A\boldsymbol{\varphi} = xB\boldsymbol{\varphi}$ to the eigenvalue problem $L^{-1}AL^{-\mathrm{T}}(L^{\mathrm{T}}\boldsymbol{\varphi}) = x(L^{\mathrm{T}}\boldsymbol{\varphi})$ and solves the eigenvalue problem. We should remark that, even if A and B are both tridiagonal, $L^{-1}AL^{-\mathrm{T}}$ is a dense matrix in general. Hence, we expect that DSYGV spends much time for large problems. On the other hand, the proposed algorithm preserves the tridiagonal form of the matrices $A^{(t)}$ and $B^{(t)}$. The proposed algorithm will thus compute the generalized eigenvalues of tridiagonal matrix pencils fast and accurately for large problems.

Tables 5.2 and 5.3 show the results of the computation by the proposed algorithm and DSYGV, respectively. The parameters for the proposed algorithm are $s^{(t)} = (N+2)/(N+1)$ for all $t \geq 0$ and $\kappa_n = -10000$ for all $n \geq N - 1$. In all the cases, the proposed algorithm is faster and more accurate than DSYGV. In particular, the proposed algorithm has an advantage in computation time for large problems. Remark that the techniques called deflation and splitting (if $|\beta_n^{(t)}|$ and $|\lambda_n \beta_n^{(t)}|$ become sufficiently small for some n at a time t, then the problem can be deflated or split into two problems) were not implemented in the program used for the experiments. These techniques

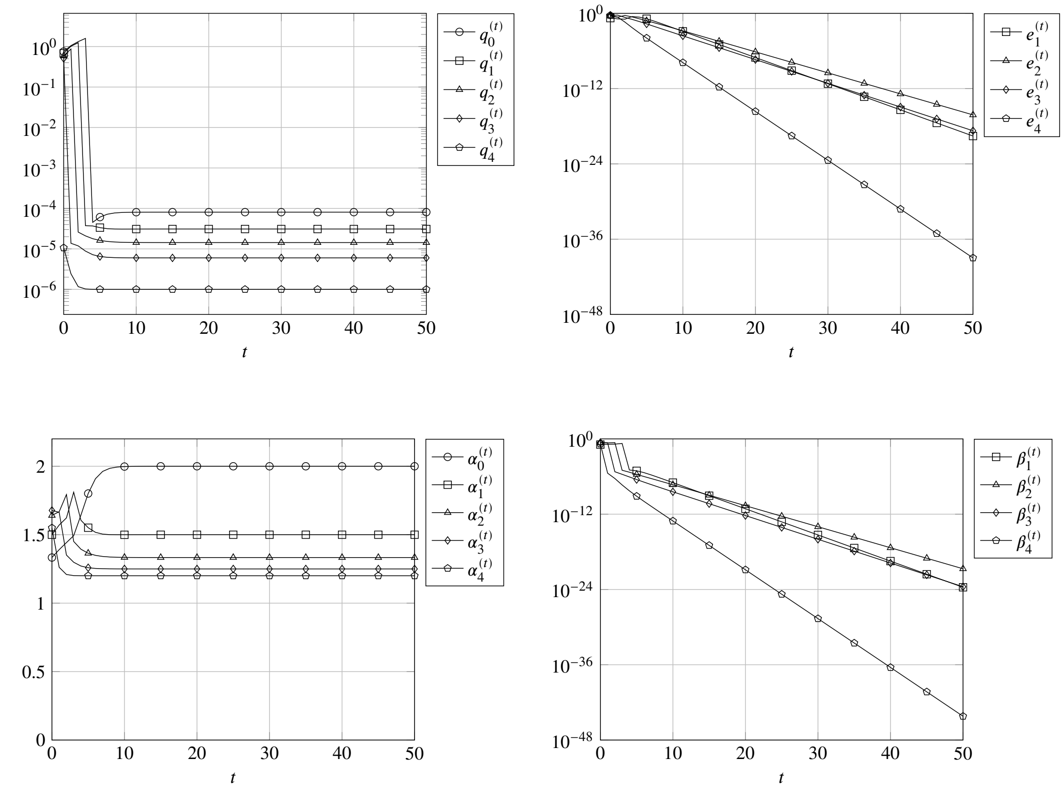


Figure 5.2: The behaviour of the variables of the monic type finite R_{II} chain for the input tridiagonal matrix pencil $(K_5 + 2I_5, K_5 + I_5)$ with the parameters $s^{(t)} = 1.19$ for all $t \geq 0$ and $\kappa_n = -10000$ for all $n \geq 4$.

may further accelerate the proposed algorithm.

Table 5.2: The results of the computation by Algorithm 3 for the generalized eigenvalue problems of $(K_N + 2I_N, K_N + I_N)$.

Problem size (N)	512	1024	2048	4096	8192
Computation time [sec.]	0.0958	0.392	1.58	6.24	24.6
Maximum relative error	3.109×10^{-15}	3.405×10^{-15}	1.776×10^{-15}	3.701×10^{-15}	2.043×10^{-14}
Average relative error	1.344×10^{-16}	1.211×10^{-16}	1.154×10^{-16}	1.072×10^{-16}	1.129×10^{-16}

Table 5.3: The results of the computation by `DSYGV` in LAPACK for the generalized eigenvalue problems of $(K_N + 2I_N, K_N + I_N)$.

Problem size (N)	512	1024	2048	4096	8192
Computation time [sec.]	0.162	1.92	30.3	307	2400
Maximum relative error	3.664×10^{-15}	6.815×10^{-15}	1.304×10^{-14}	1.684×10^{-14}	2.949×10^{-14}
Average relative error	6.673×10^{-16}	8.469×10^{-16}	1.035×10^{-15}	1.276×10^{-15}	1.508×10^{-15}

Chapter 6

R_{II} Chain and Nonautonomous Discrete Modified KdV Lattice

In this chapter, we extend a Miura type transformation between the nd-Toda lattice and the nd-LV lattice via orthogonal polynomials to the case of R_{II} polynomials.

6.1 Symmetric orthogonal polynomials and the nonautonomous discrete Lotka–Volterra Lattice

Let $\{\phi_n^{k,t}(x)\}_{n=0}^{\infty}$ be the monic orthogonal polynomials discussed in Section 3.3. We consider the polynomial sequence $\{\sigma_n^{k,t}(x)\}_{n=0}^{\infty}$ defined by

$$\sigma_{2n+i}^{k,t}(x) := x^i \phi_n^{k+i,t}(x^2), \quad n = 0, 1, 2, \ldots, \quad i = 0, 1.$$

By definition, $\sigma_n^{k,t}(x)$ is a monic polynomial of degree n and has the symmetry property

$$\sigma_n^{k,t}(-x) = (-1)^n \sigma_n^{k,t}(x).$$

Furthermore, $\{\sigma_n^{k,t}(x)\}_{n=0}^{\infty}$ are orthogonal with respect to the linear functional $\mathscr{S}^{k,t}$ defined by

$$\mathscr{S}^{k,t}[x^{2m}] := \mathscr{L}^{k,t}[x^m], \quad \mathscr{S}^{k,t}[x^{2m+1}] := 0, \quad m = 0, 1, 2, \ldots.$$

$\{\sigma_n^{k,t}(x)\}_{n=0}^{\infty}$ are called *monic symmetric orthogonal polynomials*.

From (3.9), we have the relations

$$x^2 \phi_n^{k+1,t}(x^2) = \phi_{n+1}^{k,t}(x^2) + q_n^{k,t} \phi_n^{k,t}(x^2),$$
$$x \phi_n^{k,t}(x^2) = x \phi_n^{k+1,t}(x^2) + e_n^{k,t} x \phi_{n-1}^{k+1,t}(x^2).$$

These relations lead us to the three-term recurrence relation that $\{\sigma_n^{k,t}(x)\}_{n=0}^{\infty}$ satisfy:

$$\sigma_{2n+2}^{k,t}(x) = x \sigma_{2n+1}^{k,t}(x) - q_n^{k,t} \sigma_{2n}^{k,t}(x), \tag{6.1a}$$
$$\sigma_{2n+1}^{k,t}(x) = x \sigma_{2n}^{k,t}(x) - e_n^{k,t} \sigma_{2n-1}^{k,t}(x). \tag{6.1b}$$

Spectral transformations for $\{\sigma_n^{k,t}(x)\}_{n=0}^{\infty}$ are also induced from (3.12):

$$(x^2 + s^{(t)}) \sigma_{2n}^{k,t+1}(x) = \sigma_{2n+2}^{k,t}(x) + \tilde{q}_n^{k,t} \sigma_{2n}^{k,t}(x), \tag{6.2a}$$
$$(x^2 + s^{(t)}) \sigma_{2n+1}^{k,t+1}(x) = \sigma_{2n+3}^{k,t}(x) + \tilde{q}_n^{k+1,t} \sigma_{2n+1}^{k,t}(x), \tag{6.2b}$$
$$\sigma_{2n}^{k,t}(x) = \sigma_{2n}^{k,t+1}(x) + \tilde{e}_n^{k,t} \sigma_{2n-2}^{k,t+1}(x), \tag{6.2c}$$

$$\sigma_{2n+1}^{k,t}(x) = \sigma_{2n+1}^{k,t+1}(x) + \tilde{e}_n^{k+1,t}\sigma_{2n-1}^{k,t+1}(x). \tag{6.2d}$$

In this section, we assume that the parameter $s^{(t)}$ are not zero for all t. Relations (6.1) and (6.2) show that there exist variables $v_n^{k,t}$ satisfying the relations

$$(x^2 + s^{(t)})\sigma_n^{k,t+1}(x) = x\sigma_{n+1}^{k,t}(x) + (s^{(t)} + v_n^{k,t})\sigma_n^{k,t}(x), \tag{6.3a}$$

$$(s^{(t)} + v_n^{k,t})\sigma_n^{k,t}(x) = s^{(t)}\sigma_n^{k,t+1}(x) + v_n^{k,t}x\sigma_{n-1}^{k,t+1}(x). \tag{6.3b}$$

Relations (6.3) yield

$$\begin{aligned} \sigma_{n+1}^{k,t}(x) &= x\sigma_n^{k,t}(x) - v_n^{k,t}(1 + (s^{(t)})^{-1}v_{n-1}^{k,t})\sigma_{n-1}^{k,t}(x) \\ &= x\sigma_n^{k,t}(x) - v_n^{k,t-1}(1 + (s^{(t-1)})^{-1}v_{n+1}^{k,t-1})\sigma_{n-1}^{k,t}(x). \end{aligned} \tag{6.4}$$

Hence, the compatibility condition

$$v_n^{k,t}(1 + (s^{(t)})^{-1}v_{n-1}^{k,t}) = v_n^{k,t-1}(1 + (s^{(t-1)})^{-1}v_{n+1}^{k,t-1}), \tag{6.5a}$$

$$v_0^{k,t} = 0 \quad \text{for all } k \text{ and } t, \tag{6.5b}$$

must be satisfied. This is the time evolution equation of the nd-LV lattice, a nonautonomous version of the d-LV lattice (1.11).

From relations (6.1)–(6.4), we obtain the Miura type transformation between the nd-Toda lattice (3.15) and the nd-LV lattice (6.5):

$$\begin{aligned} q_n^{k,t} &= v_{2n+1}^{k,t}(1 + (s^{(t)})^{-1}v_{2n}^{k,t}) = v_{2n+1}^{k,t-1}(1 + (s^{(t-1)})^{-1}v_{2n+2}^{k,t-1}), \\ e_n^{k,t} &= v_{2n}^{k,t}(1 + (s^{(t)})^{-1}v_{2n-1}^{k,t}) = v_{2n}^{k,t-1}(1 + (s^{(t-1)})^{-1}v_{2n+1}^{k,t-1}), \\ \tilde{q}_n^{k,t} &= s^{(t)}(1 + (s^{(t)})^{-1}v_{2n+1}^{k,t})(1 + (s^{(t)})^{-1}v_{2n}^{k,t}) \\ &= s^{(t)}(1 + (s^{(t)})^{-1}v_{2n+1}^{k-1,t})(1 + (s^{(t)})^{-1}v_{2n+2}^{k-1,t}), \\ \tilde{e}_n^{k,t} &= (s^{(t)})^{-1}v_{2n}^{k,t}v_{2n-1}^{k,t} = (s^{(t)})^{-1}v_{2n}^{k-1,t}v_{2n+1}^{k-1,t}. \end{aligned}$$

In addition, from relations (6.3), we obtain

$$\begin{aligned} 1 + (s^{(t)})^{-1}v_{2n}^{k,t} &= \frac{\sigma_{2n}^{k,t+1}(0)}{\sigma_{2n}^{k,t}(0)} = \frac{\phi_n^{k,t+1}(0)}{\phi_n^{k,t}(0)}, \\ 1 + (s^{(t)})^{-1}v_{2n-1}^{k,t} &= (-s^{(t)})^{-1/2}\frac{\sigma_{2n}^{k,t}((-s^{(t)})^{1/2})}{\sigma_{2n-1}^{k,t}((-s^{(t)})^{1/2})} = -(s^{(t)})^{-1}\frac{\phi_n^{k,t}(-s^{(t)})}{\phi_{n-1}^{k+1,t}(-s^{(t)})}. \end{aligned}$$

By using these relations and the solutions to the nd-Toda lattice (3.18), we obtain Hankel determinant solutions to the nd-LV lattice (6.5):

$$\begin{aligned} v_{2n+1}^{k,t} &= q_n^{k,t}\frac{\phi_n^{k,t}(0)}{\phi_n^{k,t+1}(0)} = \frac{\tau_n^{k,t+1}\tau_{n+1}^{k+1,t}}{\tau_{n+1}^{k,t}\tau_n^{k+1,t+1}}, \\ v_{2n}^{k,t} &= -s^{(t)}e_n^{k,t}\frac{\phi_{n-1}^{k+1,t}(-s^{(t)})}{\phi_n^{k,t}(-s^{(t)})} = s^{(t)}\frac{\tau_{n+1}^{k,t}\tau_{n-1}^{k+1,t+1}}{\tau_n^{k,t+1}\tau_n^{k+1,t}}. \end{aligned}$$

6.2 Symmetric $\mathrm{R_{II}}$ polynomials and the nonautonomous discrete modified KdV lattice

We will apply the framework constructed in the previous section to the monic $\mathrm{R_{II}}$ polynomials and derive the nd-mKdV lattice. Let us consider the monic $\mathrm{R_{II}}$ polynomials $\{\varphi_n^{k,t}(x)\}_{n=0}^{\infty}$ defined by the three-term recurrence relation of the form

$$\begin{aligned}
&\varphi_{-1}^{k,t}(x) := 0, \quad \varphi_0^{k,t}(x) := 1, \\
&\varphi_{n+1}^{k,t}(x) := ((1+\beta_n^{k,t})x - \alpha_n^{k,t})\varphi_n^{k,t}(x) - \beta_n^{k,t}(x+\gamma_{k+t+2n-2})(x+\gamma_{k+t+2n-1})\varphi_{n-1}^{k,t}(x), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 0, 1, 2, \dots,
\end{aligned}$$

where $\alpha_n^{k,t} \in \mathbb{R}$ and $\beta_n^{k,t}, \gamma_{k+t+n} \in \mathbb{R} - \{0\}$. If nonzero constants $h_0^{k,t}$ and $h_1^{k,t}$ are fixed, then a Favard type theorem [34] guarantees the existence of a unique linear functional $\mathscr{L}^{k,t}$ such that the orthogonality relation

$$\mathscr{L}^{k,t}\left[\frac{x^m \varphi_n^{k,t}(x)}{\prod_{j=0}^{2n-1}(x+\gamma_{k+t+j})}\right] = h_n^{k,t}\delta_{m,n}, \quad n = 0, 1, 2, \dots, \quad m = 0, 1, \dots, n,$$

holds, where $h_n^{k,t}$, $n = 2, 3, \dots$ are nonzero constants. Note that $\mathscr{L}^{k,t}$ is defined on the vector space spanned by $\frac{1}{\prod_{j=0}^{l-1}(x+\gamma_{k+t+j})}$, $l = 0, 1, 2, \dots$.

As in the case of monic orthogonal polynomials, we introduce the time evolution of the monic $\mathrm{R_{II}}$ polynomials by the following spectral transformations:

$$(1+q_n^{k,t})x\varphi_n^{k+1,t}(x) = \varphi_{n+1}^{k,t}(x) + q_n^{k,t}(x+\gamma_{k+t+2n})\varphi_n^{k,t}(x), \tag{6.6a}$$

$$(1+e_n^{k,t})\varphi_n^{k,t}(x) = \varphi_n^{k+1,t}(x) + e_n^{k,t}(x+\gamma_{k+t+2n-1})\varphi_{n-1}^{k+1,t}(x), \tag{6.6b}$$

$$(1+\tilde{q}_n^{k,t})(x+s^{(t)})\varphi_n^{k,t+1}(x) = \varphi_{n+1}^{k,t}(x) + \tilde{q}_n^{k,t}(x+\gamma_{k+t+2n})\varphi_n^{k,t}(x), \tag{6.6c}$$

$$(1+\tilde{e}_n^{k,t})\varphi_n^{k,t}(x) = \varphi_n^{k,t+1}(x) + \tilde{e}_n^{k,t}(x+\gamma_{k+t+2n-1})\varphi_{n-1}^{k,t+1}(x). \tag{6.6d}$$

The time evolution of the linear functional is also given by

$$\mathscr{L}^{k+1,t}[\rho(x)] := \mathscr{L}^{k,t}\left[\frac{x}{x+\gamma_{k+t}}\rho(x)\right], \quad \mathscr{L}^{k,t+1}[\rho(x)] := \mathscr{L}^{k,t}\left[\frac{x+s^{(t)}}{x+\gamma_{k+t}}\rho(x)\right] \tag{6.7}$$

for rational functions $\rho(x)$. One can verify that $\{\varphi_n^{k+1,t}(x)\}_{n=0}^{\infty}$ and $\{\varphi_n^{k,t+1}(x)\}_{n=0}^{\infty}$ are both also monic $\mathrm{R_{II}}$ polynomials. The spectral transformations (6.6) induce the time evolution equations of the monic type $\mathrm{R_{II}}$ chain:

$$\begin{aligned}
\alpha_n^{k,t} &= \gamma_{k+t+2n}q_n^{k,t} + \gamma_{k+t+2n-1}e_n^{k,t}\frac{1+q_n^{k,t}}{1+q_{n-1}^{k,t}} \\
&= \gamma_{k+t+2n-1}q_n^{k-1,t}\frac{1+e_{n+1}^{k-1,t}}{1+e_n^{k-1,t}} + \gamma_{k+t+2n}e_{n+1}^{k-1,t} \\
&= \gamma_{k+t+2n}\tilde{q}_n^{k,t} + \gamma_{k+t+2n-1}\tilde{e}_n^{k,t}\frac{1+\tilde{q}_n^{k,t}}{1+\tilde{q}_{n-1}^{k,t}} - s^{(t)}(1+\tilde{q}_n^{k,t})(1+\tilde{e}_n^{k,t}) \\
&= \gamma_{k+t+2n-1}\tilde{q}_n^{k,t-1}\frac{1+\tilde{e}_{n+1}^{k,t-1}}{1+\tilde{e}_n^{k,t-1}} + \gamma_{k+t+2n}\tilde{e}_{n+1}^{k,t-1} - s^{(t-1)}(1+\tilde{q}_n^{k,t-1})(1+\tilde{e}_{n+1}^{k,t-1}),
\end{aligned} \tag{6.8a}$$

$$
\begin{aligned}
\beta_n^{k,t} &= q_{n-1}^{k,t} e_n^{k,t} \frac{1+q_n^{k,t}}{1+q_{n-1}^{k,t}} = q_n^{k-1,t} e_n^{k-1,t} \frac{1+e_{n+1}^{k-1,t}}{1+e_n^{k-1,t}} \\
&= \tilde{q}_{n-1}^{k,t} \tilde{e}_n^{k,t} \frac{1+\tilde{q}_n^{k,t}}{1+\tilde{q}_{n-1}^{k,t}} = \tilde{q}_n^{k,t-1} \tilde{e}_n^{k,t-1} \frac{1+\tilde{e}_{n+1}^{k,t-1}}{1+\tilde{e}_n^{k,t-1}},
\end{aligned} \tag{6.8b}
$$

$$
e_0^{k,t} = \tilde{e}_0^{k,t} = 0 \quad \text{for all } k \text{ and } t. \tag{6.8c}
$$

We define the moment of the linear functional $\mathscr{L}^{0,t}$ by

$$
\mu_{m,l}^{(t)} := \mathscr{L}^{0,t}\left[\frac{x^m}{\prod_{j=0}^{l-1}(x+\gamma_{t+j})}\right].
$$

Note that the time evolution of the linear functional (6.7) gives the relation

$$
\mathscr{L}^{k,t}\left[\frac{x^m}{\prod_{j=0}^{l-1}(x+\gamma_{k+t+j})}\right] = \mathscr{L}^{0,t}\left[\frac{x^{k+m}}{\prod_{j=0}^{k+l-1}(x+\gamma_{t+j})}\right] = \mu_{k+m,k+l}^{(t)}
$$

and the dispersion relations

$$
\mu_{m,l}^{(t+1)} = \mu_{m+1,l+1}^{(t)} + s^{(t)}\mu_{m,l+1}^{(t)}, \quad \mu_{m,l}^{(t)} = \mu_{m+1,l+1}^{(t)} + \gamma_{t+l}\mu_{m,l+1}^{(t)}. \tag{6.9}
$$

Then, the determinant expression of the monic $\mathrm{R_{II}}$ polynomials $\{\varphi_n^{k,t}(x)\}_{n=0}^{\infty}$ is given by

$$
\varphi_n^{k,t}(x) = \frac{1}{\tau_n^{k,2n,t}} \begin{vmatrix}
\mu_{k,k+2n}^{(t)} & \mu_{k+1,k+2n}^{(t)} & \cdots & \mu_{k+n-1,k+2n}^{(t)} & \mu_{k+n,k+2n}^{(t)} \\
\mu_{k+1,k+2n}^{(t)} & \mu_{k+2,k+2n}^{(t)} & \cdots & \mu_{k+n,k+2n}^{(t)} & \mu_{k+n+1,k+2n}^{(t)} \\
\vdots & \vdots & & \vdots & \vdots \\
\mu_{k+n-1,k+2n}^{(t)} & \mu_{k+n,k+2n}^{(t)} & \cdots & \mu_{k+2n-2,k+2n}^{(t)} & \mu_{k+2n-1,k+2n}^{(t)} \\
1 & x & \cdots & x^{n-1} & x^n
\end{vmatrix}, \tag{6.10}
$$

where $\tau_n^{k,l,t}$ is the Hankel determinant of order n:

$$
\tau_{-1}^{k,l,t} := 0, \quad \tau_0^{k,l,t} := 1, \quad \tau_n^{k,l,t} := |\mu_{k+i+j,k+l}^{(t)}|_{0\le i,j\le n-1}, \quad n = 1,2,3,\dots. \tag{6.11}
$$

By using the determinant expression (6.10), the dispersion relation (6.9), and a determinant identity called the Plücker relation, we can find the Hankel determinant solutions to the monic type $\mathrm{R_{II}}$ chain:

$$
\begin{aligned}
q_n^{k,t} &= (\gamma_{k+t+2n})^{-1}\frac{\tau_n^{k,2n,t}\tau_{n+1}^{k+1,2n+1,t}}{\tau_{n+1}^{k,2n+2,t}\tau_n^{k+1,2n-1,t}}, & e_n^{k,t} &= \gamma_{k+t+2n}\frac{\tau_{n+1}^{k,2n+1,t}\tau_{n-1}^{k+1,2n-2,t}}{\tau_n^{k,2n-1,t}\tau_n^{k+1,2n,t}}, \\
\tilde{q}_n^{k,t} &= (\gamma_{k+t+2n}-s^{(t)})^{-1}\frac{\tau_n^{k,2n,t}\tau_{n+1}^{k,2n+1,t+1}}{\tau_{n+1}^{k,2n+2,t}\tau_n^{k,2n-1,t+1}}, & \tilde{e}_n^{k,t} &= (\gamma_{k+t+2n}-s^{(t)})\frac{\tau_{n+1}^{k,2n+1,t}\tau_{n-1}^{k,2n-2,t+1}}{\tau_n^{k,2n-1,t}\tau_n^{k,2n,t+1}}.
\end{aligned}
$$

Next, we introduce a symmetric version of the monic $\mathrm{R_{II}}$ polynomials, which is an analogue of the monic symmetric orthogonal polynomials. Let us define a polynomial sequence $\{\varsigma_n^{k,t}(x)\}_{n=0}^{\infty}$ by

$$
\varsigma_{2n+i}^{k,t}(x) := x^i\varphi_n^{k+i,t}(x^2), \quad n = 0,1,2,\dots, \quad i = 0,1.
$$

The corresponding linear functional $\mathscr{S}^{k,t}$ is given by

$$\mathscr{S}^{k,t}\left[\frac{x^{2m}}{\prod_{j=0}^{l-1}(x^2+\gamma_{k+t+j})}\right] := \mathscr{L}^{k,t}\left[\frac{x^{m}}{\prod_{j=0}^{l-1}(x+\gamma_{k+t+j})}\right],\ \mathscr{S}^{k,t}\left[\frac{x^{2m+1}}{\prod_{j=0}^{l-1}(x^2+\gamma_{k+t+j})}\right] := 0,$$
$$l=0,1,2,\dots,\quad m=0,1,\dots,l.$$

The spectral transformations (6.6) yield the three-term recurrence relations

$$\varsigma_{2n+2}^{k,t}(x) = (1+q_n^{k,t})x\varsigma_{2n+1}^{k,t}(x) - q_n^{k,t}(x^2+\gamma_{k+t+2n})\varsigma_{2n}^{k,t}(x), \tag{6.12a}$$
$$\varsigma_{2n+1}^{k,t}(x) = (1+e_n^{k,t})x\varsigma_{2n}^{k,t}(x) - e_n^{k,t}(x^2+\gamma_{k+t+2n-1})\varsigma_{2n-1}^{k,t}(x), \tag{6.12b}$$

and the spectral transformations for the monic symmetric $\mathrm{R_{II}}$ polynomials $\{\varsigma_n^{k,t}(x)\}_{n=0}^{\infty}$:

$$(1+\tilde{q}_n^{k,t})(x^2+s^{(t)})\varsigma_{2n}^{k,t+1}(x) = \varsigma_{2n+2}^{k,t}(x) + \tilde{q}_n^{k,t}(x^2+\gamma_{k+t+2n})\varsigma_{2n}^{k,t}(x), \tag{6.13a}$$
$$(1+\tilde{q}_n^{k+1,t})(x^2+s^{(t)})\varsigma_{2n+1}^{k,t+1}(x) = \varsigma_{2n+3}^{k,t}(x) + \tilde{q}_n^{k+1,t}(x^2+\gamma_{k+t+2n+1})\varsigma_{2n+1}^{k,t}(x), \tag{6.13b}$$
$$(1+\tilde{e}_n^{k,t})\varsigma_{2n}^{k,t}(x) = \varsigma_{2n}^{k,t+1}(x) + \tilde{e}_n^{k,t}(x^2+\gamma_{k+t+2n-1})\varsigma_{2n-2}^{k,t+1}(x), \tag{6.13c}$$
$$(1+\tilde{e}_n^{k+1,t})\varsigma_{2n+1}^{k,t}(x) = \varsigma_{2n+1}^{k,t+1}(x) + \tilde{e}_n^{k+1,t}(x^2+\gamma_{k+t+2n})\varsigma_{2n-1}^{k,t+1}(x). \tag{6.13d}$$

Relations (6.12) and (6.13) show that there exist variables $v_n^{k,t}$ satisfying the relations

$$\begin{aligned}&(\gamma_{k+t+n}+v_n^{k,t})(x^2+s^{(t)})\varsigma_n^{k,t+1}(x)\\ &\qquad = (\gamma_{k+t+n}-s^{(t)})x\varsigma_{n+1}^{k,t}(x) + (s^{(t)}+v_n^{k,t})(x^2+\gamma_{k+t+n})\varsigma_n^{k,t}(x),\end{aligned} \tag{6.14a}$$
$$\begin{aligned}&\gamma_{k+t+n}(s^{(t)}+v_n^{k,t})\varsigma_n^{k,t}(x)\\ &\qquad = s^{(t)}(\gamma_{k+t+n}+v_n^{k,t})\varsigma_n^{k,t+1}(x) + (\gamma_{k+t+n}-s^{(t)})v_n^{k,t}x\varsigma_{n-1}^{k,t+1}(x).\end{aligned} \tag{6.14b}$$

Relations (6.14) yield

$$\begin{aligned}\varsigma_{n+1}^{k,t}(x) &= \left(1+(\gamma_{k+t+n-1})^{-1}v_n^{k,t}\frac{1+(s^{(t)})^{-1}v_{n-1}^{k,t}}{1+(\gamma_{k+t+n-1})^{-1}v_{n-1}^{k,t}}\right)x\varsigma_n^{k,t}(x)\\ &\qquad - (\gamma_{k+t+n-1})^{-1}v_n^{k,t}\frac{1+(s^{(t)})^{-1}v_{n-1}^{k,t}}{1+(\gamma_{k+t+n-1})^{-1}v_{n-1}^{k,t}}(x^2+\gamma_{k+t+n-1})\varsigma_{n-1}^{k,t}(x)\\ &= \left(1+(\gamma_{k+t+n-1})^{-1}v_n^{k,t-1}\frac{1+(s^{(t-1)})^{-1}v_{n+1}^{k,t-1}}{1+(\gamma_{k+t+n})^{-1}v_{n+1}^{k,t-1}}\right)x\varsigma_n^{k,t}(x)\\ &\qquad - (\gamma_{k+t+n-1})^{-1}v_n^{k,t-1}\frac{1+(s^{(t-1)})^{-1}v_{n+1}^{k,t-1}}{1+(\gamma_{k+t+n})^{-1}v_{n+1}^{k,t-1}}(x^2+\gamma_{k+t+n-1})\varsigma_{n-1}^{k,t}(x).\end{aligned}$$

Hence, the compatibility condition

$$v_n^{k,t}\frac{1+(s^{(t)})^{-1}v_{n-1}^{k,t}}{1+(\gamma_{k+t+n-1})^{-1}v_{n-1}^{k,t}} = v_n^{k,t-1}\frac{1+(s^{(t-1)})^{-1}v_{n+1}^{k,t-1}}{1+(\gamma_{k+t+n})^{-1}v_{n+1}^{k,t-1}}, \tag{6.15a}$$
$$v_0^{k,t} = 0 \quad \text{for all } k \text{ and } t, \tag{6.15b}$$

must be satisfied. This is the time evolution equation of the nd-mKdV lattice, a nonautonomous version of the discrete mKdV lattice [76]. Note that the nd-mKdV lattice (6.15) reduces to the nd-LV lattice (6.5) as $\gamma_{k+t+n}\to\infty$.

The Miura type transformation between the monic type $\mathrm{R_{II}}$ chain (6.8) and the nd-mKdV lattice (6.15) is obtained as follows:

$$\begin{aligned}
q_n^{k,t} &= (\gamma_{k+t+2n})^{-1} v_{2n+1}^{k,t} \frac{1 + (s^{(t)})^{-1} v_{2n}^{k,t}}{1 + (\gamma_{k+t+2n})^{-1} v_{2n}^{k,t}} \\
&= (\gamma_{k+t+2n})^{-1} v_{2n+1}^{k,t-1} \frac{1 + (s^{(t-1)})^{-1} v_{2n+2}^{k,t-1}}{1 + (\gamma_{k+t+2n+1})^{-1} v_{2n+2}^{k,t-1}}, \\
e_n^{k,t} &= (\gamma_{k+t+2n-1})^{-1} v_{2n}^{k,t} \frac{1 + (s^{(t)})^{-1} v_{2n-1}^{k,t}}{1 + (\gamma_{k+t+2n-1})^{-1} v_{2n-1}^{k,t}} \\
&= (\gamma_{k+t+2n-1})^{-1} v_{2n}^{k,t-1} \frac{1 + (s^{(t-1)})^{-1} v_{2n+1}^{k,t-1}}{1 + (\gamma_{k+t+2n})^{-1} v_{2n+1}^{k,t-1}}, \\
\tilde{q}_n^{k,t} &= (\gamma_{k+t+2n} - s^{(t)})^{-1} s^{(t)} \big(1 + (s^{(t)})^{-1} v_{2n+1}^{k,t}\big)\big(1 + (s^{(t)})^{-1} v_{2n}^{k,t}\big) \\
&= (\gamma_{k+t+2n} - s^{(t)})^{-1} s^{(t)} \big(1 + (s^{(t)})^{-1} v_{2n+1}^{k-1,t}\big)\big(1 + (s^{(t)})^{-1} v_{2n+2}^{k-1,t}\big), \\
\tilde{e}_n^{k,t} &= \frac{(\gamma_{k+t+2n} - s^{(t)}) v_{2n}^{k,t} v_{2n-1}^{k,t}}{s^{(t)} \gamma_{k+t+2n-1} \gamma_{k+t+2n} \big(1 + (\gamma_{k+t+2n})^{-1} v_{2n}^{k,t}\big)\big(1 + (\gamma_{k+t+2n-1})^{-1} v_{2n-1}^{k,t}\big)} \\
&= \frac{(\gamma_{k+t+2n} - s^{(t)}) v_{2n}^{k-1,t} v_{2n+1}^{k-1,t}}{s^{(t)} \gamma_{k+t+2n-1} \gamma_{k+t+2n} \big(1 + (\gamma_{k+t+2n-1})^{-1} v_{2n}^{k-1,t}\big)\big(1 + (\gamma_{k+t+2n})^{-1} v_{2n+1}^{k-1,t}\big)}.
\end{aligned}$$

Furthermore, from relations (6.14), we obtain

$$\begin{aligned}
&\frac{1 + (s^{(t)})^{-1} v_{2n}^{k,t}}{1 + (\gamma_{k+t+2n})^{-1} v_{2n}^{k,t}} = \frac{\varphi_n^{k,t+1}(0)}{\varphi_n^{k,t}(0)}, \\
&1 + (s^{(t)})^{-1} v_{2n-1}^{k,t} = (-s^{(t)})^{-1} \frac{\varphi_n^{k,t}(-s^{(t)})}{\varphi_{n-1}^{k+1,t}(-s^{(t)})}, \\
&1 + (\gamma_{k+t+2n-1})^{-1} v_{2n-1}^{k,t} = (-\gamma_{k+t+2n-1})^{-1} \frac{\varphi_n^{k,t}(-\gamma_{k+t+2n-1})}{\varphi_{n-1}^{k+1,t+1}(-\gamma_{k+t+2n-1})}.
\end{aligned}$$

Hence, Hankel determinant solutions to the nd-mKdV lattice (6.15) are given by

$$\begin{aligned}
v_{2n+1}^{k,t} &= \gamma_{k+t+2n} q_n^{k,t} \frac{\varphi_n^{k,t}(0)}{\varphi_n^{k,t+1}(0)} = \frac{\tau_n^{k,2n,t+1} \tau_{n+1}^{k+1,2n+1,t}}{\tau_{n+1}^{k,2n+2,t} \tau_n^{k+1,2n-1,t+1}}, \\
v_{2n}^{k,t} &= s^{(t)} e_n^{k,t} \frac{\varphi_{n-1}^{k+1,t}(-s^{(t)})}{\varphi_n^{k,t}(-s^{(t)})} \frac{\varphi_n^{k,t}(-\gamma_{k+t+2n-1})}{\varphi_{n-1}^{k+1,t+1}(-\gamma_{k+t+2n-1})} = s^{(t)} \gamma_{k+t+2n} \frac{\tau_{n+1}^{k,2n+1,t} \tau_{n-1}^{k+1,2n-2,t+1}}{\tau_n^{k,2n-1,t+1} \tau_n^{k+1,2n,t}}.
\end{aligned}$$

Remark 6.1. Spiridonov [67] first considered spectral transformations for the (not monic) symmetric $\mathrm{R_{II}}$ polynomials. By using spectral transformations, he derived a generalization of the nd-LV lattice (6.5), which is more complicated than the nd-mKdV lattice (6.15). We have considered the monic symmetric $\mathrm{R_{II}}$ polynomials and their spectral transformations which possess the following symmetry. Consider an independent variable transformation $t' = -k - t - n$ and introduce $\tilde{\zeta}_n^{k,t'}(x) := \zeta_n^{k,-k-t'-n+1}(x)$, $\tilde{v}_n^{k,t'} := v_n^{k,-k-t'-n}$, $\tilde{s}_{k+t'+n} := s^{(-k-t'-n)}$ and $\tilde{\gamma}^{(t')} := \gamma_{-t'}$. Then, the spectral transformations for the monic symmetric $\mathrm{R_{II}}$ polynomials (6.14) may be rewritten as

$$\begin{aligned}
&(\tilde{s}_{k+t'+n} + \tilde{v}_n^{k,t'})(x^2 + \tilde{\gamma}^{(t')})\tilde{\zeta}_n^{k,t'+1}(x) \\
&\qquad = (\tilde{s}_{k+t'+n} - \tilde{\gamma}^{(t')}) x \tilde{\zeta}_{n+1}^{k,t'}(x) + (\tilde{\gamma}^{(t')} + \tilde{v}_n^{k,t'})(x^2 + \tilde{s}_{k+t'+n})\tilde{\zeta}_n^{k,t'}(x),
\end{aligned}$$

$$
\begin{aligned}
&\tilde{s}_{k+t'+n}(\tilde{\gamma}^{(t')} + \tilde{v}_n^{k,t'})\tilde{\zeta}_n^{k,t'}(x) \\
&\qquad = \tilde{\gamma}^{(t')}(\tilde{s}_{k+t'+n} + \tilde{v}_n^{k,t'})\tilde{\zeta}_n^{k,t'+1}(x) + (\tilde{s}_{k+t'+n} - \tilde{\gamma}^{(t')})\tilde{v}_n^{k,t'} x\tilde{\zeta}_{n-1}^{k,t'+1}(x),
\end{aligned}
$$

so that the roles of the parameters are replaced. Using the symmetric form of the spectral transformations (6.14), we can derive the corresponding discrete integrable lattice in a simpler form.

In another study, Spiridonov *et al.* [69] derived a discrete integrable lattice called the FST chain and discussed its connection to the $\mathrm{R_{II}}$ chain. The time evolution equation of the FST chain is

$$
\begin{aligned}
&\frac{\gamma_{k+t+n} - s^{(t)} + A_n^{k,t} A_{n-1}^{k,t}}{A_n^{k,t}} = \frac{\gamma_{k+t+n-1} - s^{(t-1)} + A_n^{k,t-1} A_{n+1}^{k,t-1}}{A_n^{k,t-1}}, \\
&A_{-1}^{k,t} = 0 \quad \text{for all } k \text{ and } t.
\end{aligned}
$$

Particular solutions to the FST chain may also be expressed by the Hankel determinant (6.11):

$$
A_{2n}^{k,t} = (\gamma_{k+t+2n} - s^{(t)})\frac{\tau_{n+1}^{k,2n+1,t}\tau_n^{k,2n-1,t+1}}{\tau_n^{k,2n,t}\tau_n^{k,2n,t+1}}, \quad A_{2n+1}^{k,t} = (\gamma_{k+t+2n+1} - s^{(t)})\frac{\tau_{n+1}^{k,2n+2,t}\tau_n^{k,2n,t+1}}{\tau_{n+1}^{k,2n+1,t}\tau_{n+1}^{k,2n+1,t+1}}.
$$

Similarly, we have the discrete potential KdV lattice

$$
\begin{aligned}
&(\epsilon_n^{k,t-1} - \epsilon_n^{k,t})(\epsilon_{n+1}^{k,t-1} - \epsilon_{n-1}^{k,t}) = \gamma_{k+t+n-1} - s^{(t-1)}, \\
&\epsilon_{-1}^{k,t} = 0 \quad \text{for all } k \text{ and } t,
\end{aligned}
$$

and its Hankel determinant solutions

$$
\epsilon_{2n}^{k,t} = \frac{\tau_{n+1}^{k,2n,t}}{\tau_n^{k,2n,t}}, \quad \epsilon_{2n+1}^{k,t} = \frac{\tau_n^{k,2n+1,t}}{\tau_{n+1}^{k,2n+1,t}}.
$$

Therefore, these systems and the nd-mKdV lattice (6.15) are connected via the bilinear formalism.

Chapter 7

Conclusion

In this chapter, we conclude the thesis with some remarks.

In Chapter 2, we summarized the theory of monic orthogonal polynomials and the ndf-Toda lattice, whose recurrence relations are also called the dqds algorithm in numerical analysis. The nd-Toda lattice is derived as the compatibility condition for two spectral transformations of monic orthogonal polynomials, which are called the Christoffel transformation and the Geronimus transformation, respectively. For monic infinite orthogonal polynomial sequences, the corresponding lattice is the nd-Toda lattice. On the other hand, for monic finite orthogonal polynomial sequences, the corresponding lattice is the ndf-Toda lattice. Since the monic finite orthogonal polynomials have concrete determinant expressions written by the zeros of the highest degree polynomial, which are just the eigenvalues of the corresponding tridiagonal matrix, the solution to the initial value problem of the ndf-Toda lattice is also written by the eigenvalues. This theory has been a powerful framework to discuss the following topics.

In Chapter 3, we derived the modified nd-Toda lattice and its solutions from monic orthogonal polynomials. By ultradiscretizing the modified nd-Toda lattice, we clarified the correspondence between the modified nuf-Toda lattice and the BBS with carrier capacity, and presented a particular solution to the modified nuf-Toda lattice. Moreover, we discussed the relation between the BBS with carrier capacity and the dqds algorithm. As the results, the following correspondences were obtained: (i) the size of solitons and the eigenvalue of tridiagonal matrices, (ii) the distance between two solitons and the value of subdiagonal elements of tridiagonal matrices, and (iii) the carrier capacity and the origin shift parameter.

In Chapter 4, we derived the finite Toda representation of the BBS with box capacity by introducing the expansion map from a state of the BBS to a binary sequence. Furthermore, we presented a particular solution for the fixed box capacity case. Hence, we can say that the ultradiscrete system (4.13) is integrable if the parameters $K_n^{(t)}$ and $\Lambda_n^{(t)}$ are chosen as constants. Since there is a connection between the ultradiscrete system (4.13) and the BBS with variable box capacity which is integrable, we expect that the ultradiscrete system (4.13) of the variable box capacity case is also integrable and a discrete system derived through the inverse-ultradiscretization has determinant solutions. This novel discrete integrable system will give us a new type of orthogonal functions and applications to numerical algorithms.

In the proof of Theorem 4.3, the variables $\tilde{C}_n^{(t+1)}$ and $\tilde{D}_n^{(t+1)}$ played important roles; these variables denote the number of balls which the carrier has. Moreover, these variables correspond to the variables which are introduced to remove subtractions in the discrete equations. This result gives us a guideline for the ultradiscretization of discrete integrable lattices and making connections between these systems and BBSs.

In Chapter 5, we studied the monic type R_{II} chain in detail and proposed a generalized eigenvalue algorithm for tridiagonal matrix pencils based on a subtraction-free form of the monic type finite R_{II} chain. It was shown that, similarly to the dqds algorithm, the parameter $s^{(t)}$ in the monic type finite R_{II} chain plays the role of the origin shifts to accelerate convergence and the proposed algorithm computes the generalized eigenvalues of tridiagonal matrix pencils fast and accurately.

In Example 5.5, the shift parameter $s^{(t)}$ is chosen ideally and all the conditions (5.22) are satisfied. However, it is difficult to make this situation in general. Further improvements are thus required for practical use. First, in general, the condition for positivity (5.22) is not sufficient for applications; the condition does not provide concrete ways to choose the parameters for general cases. Second, for applying the proposed algorithm to general (not tridiagonal) matrix pencils, a preconditioning called simultaneous tridiagonalization (see, e.g., [15,66]) is required. In addition to the improvements, comparisons with traditional methods should be discussed.

In Chapter 6, we developed the spectral transformation technique for symmetric R_{II} polynomials and derived the nd-mKdV lattice as the compatibility condition. Moreover, we obtained a direct connection between the R_{II} chain and the nd-mKdV lattice. It is easily verified by numerical experiments that the obtained nd-mKdV lattice with a non-periodic finite lattice condition can compute the generalized eigenvalues of the tridiagonal matrix pencil that corresponds to the R_{II} polynomials through the Miura type transformation. The results will yield more practical applications of the nd-mKdV lattice to numerical algorithms, e.g., generalized singular value decomposition [88].

In recent studies, various discrete Painlevé equations have been obtained as reductions of discrete integrable systems [18,20,56–59]. On the other hand, it is known that the R_{II} chain and the elliptic Painlevé equation [62] have solutions expressible in terms of the elliptic hypergeometric function ${}_{12}V_{11}$ [37, 68, 73]. In addition, it was pointed out that the contiguity relations of the elliptic Painlevé equation are similar to the linear relations of the R_{II} chain [55]. Supported by these evidences, one may believe that a reduction of the R_{II} chain may give rise to the elliptic Painlevé equation. This work linked the nd-mKdV lattice with the R_{II} chain. We are now concerned with its relationship to the discrete Painlevé equations. In particular, we expect that the elliptic Painlevé equation will appear as a reduction of the nd-mKdV lattice.

In this thesis, we provided a unified framework to deal with discrete integrable finite lattices, BBSs and numerical algorithms through the theory of orthogonal polynomials. These objects individually have their own established fields, so that this framework allows us to import the results in one field into another field. We hope that this work contributes to the further development of these fields, or even of other fields.

Acknowledgements

The author would like to express his sincere gratitude toward his supervisor Professor Satoshi Tsujimoto for his great encouragement and suggestions. Without his dedicated support, this work could never have been accomplished. He would like to express his gratitude also to Professor Yoshimasa Nakamura for his generous support and helpful advices. He would like to express his thanks to Professors Shuhei Kamioka, Kinji Kimura, Hiroshi Miki, Hiroto Sekido and Alexei Zhedanov for their great insights and fruitful discussions. He would like to thank the members and the alumni of Applied Mathematical Analysis Laboratory at Graduate School of Informatics, Kyoto University, for valuable discussions and comments.

He gratefully acknowledges the financial support as Research Fellowship for Young Scientists from Japan Society for the Promotion of Science. Special thanks go to Ms. Masako Fujikawa and Ms. Mako Iida for secretarial assistance.

Finally, he would like to express his deep gratitude to his parents for their heartfelt encouragement and support.

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List of Authors Papers and Works Related to the Thesis

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